SLOVAK UNIVERSITY OF
TECHNOLOGY IN BRATISLAVA

European Space Agency
Agence spatiale européenne

Orbital Mechanics

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Space for Education, Education for Space
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Specialized lectures

Contents

1. The two body problem
2. Orbits in three dimensions
3. Orbital perturbations
4. Orbital maneuvers

1. The two body problem

- Motion in inertial frame
- Relative motion
- Angular momentum
- Solution of problem
- Energy law
- Trajectories
- Time and position

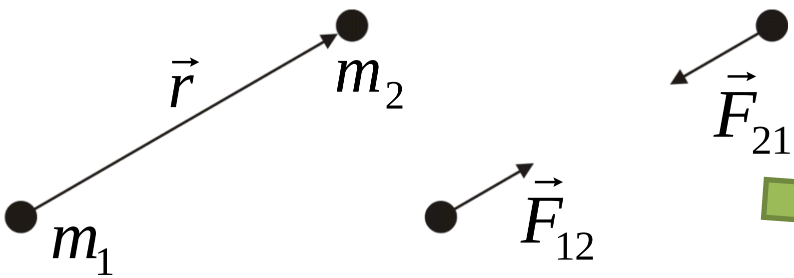
1. The two body problem

Motion in inertial frame

- Two body problem can by defined by:
  - Newton's law of gravitation

position of masses

gravitational forces



$$\vec{F}_{21} = -\vec{F}_{12} = -G\frac{m_1 m_2}{r^3}\vec{r}$$

universal gravitational constant

$$G = 6.6742\times10^{-11}\ \mathrm{m}^3/\mathrm{kg\,s}^2$$

1. The two body problem

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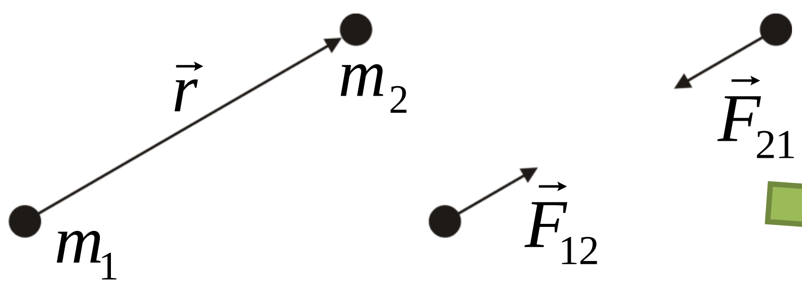

$$\vec{F}_{21} = -\vec{F}_{12} = -G\frac{m_1 m_2}{r^3}\vec{r}$$

1st time measured by Cavendish, 1798

universal gravitational constant

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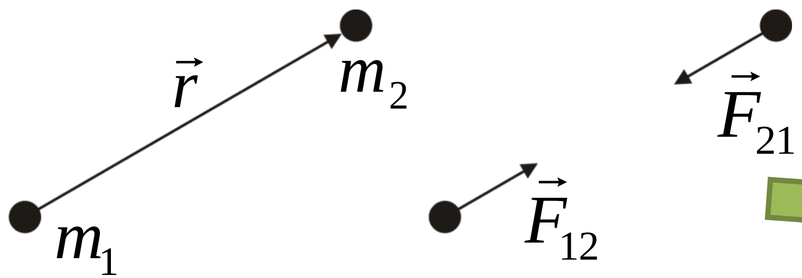

$$\vec{F}_{21} = -\vec{F}_{12} = -G\frac{m_1 m_2}{r^3}\vec{r}$$

conservative force can be expressed by potential energy

$$E_p = -G\frac{m_1 m_2}{r}$$

universal gravitational constant

$$G = 6.6742 \times 10^{-11}\ \mathrm{m}^3/\mathrm{kg\,s}^2$$

1st time measured by Cavendish, 1798

1. The two body problem

Motion in inertial frame

- Two body problem can by defined by:
  - Newton's law of gravitation

Central body	μ [m³/s²]
Earth	$3.98600441 \times 10^{14}$
Moon	$4.90279888 \times 10^{12}$
Mars	4.2871×10^{13}
Sun	1.327124×10^{20}

$$\vec{F}_{21} = -\vec{F}_{12} = -G\frac{m_1 m_2}{r^3}\vec{r}$$

μ can be measured with considerable precision by astronomical observation

$$\vec{F}_{21} = -\vec{F}_{12} = -\frac{\mu\, m_2}{r^3}\vec{r}$$

1. The two body problem

Motion in inertial frame

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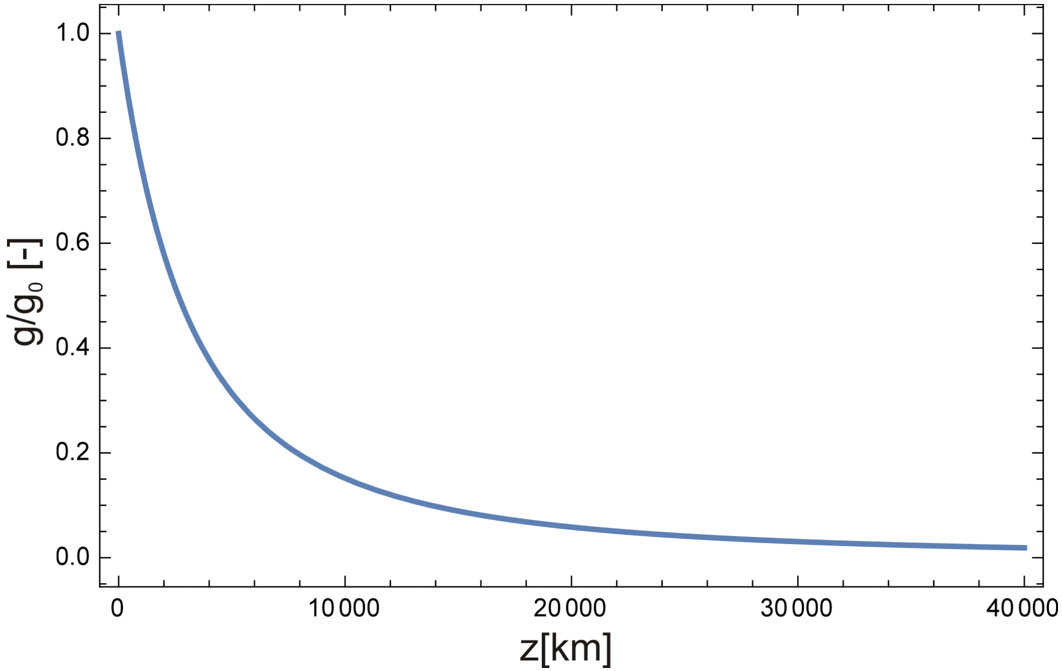

$$\vec{F}_{21} = -\vec{F}_{12} = -G\frac{m_1 m_2}{r^3}\vec{r}$$

$$g = G\frac{m_E}{r^2} = g_0\left(\frac{r_E}{r_E + z}\right)^2$$

1. The two body problem

Motion in inertial frame

- Two body problem can by defined by:
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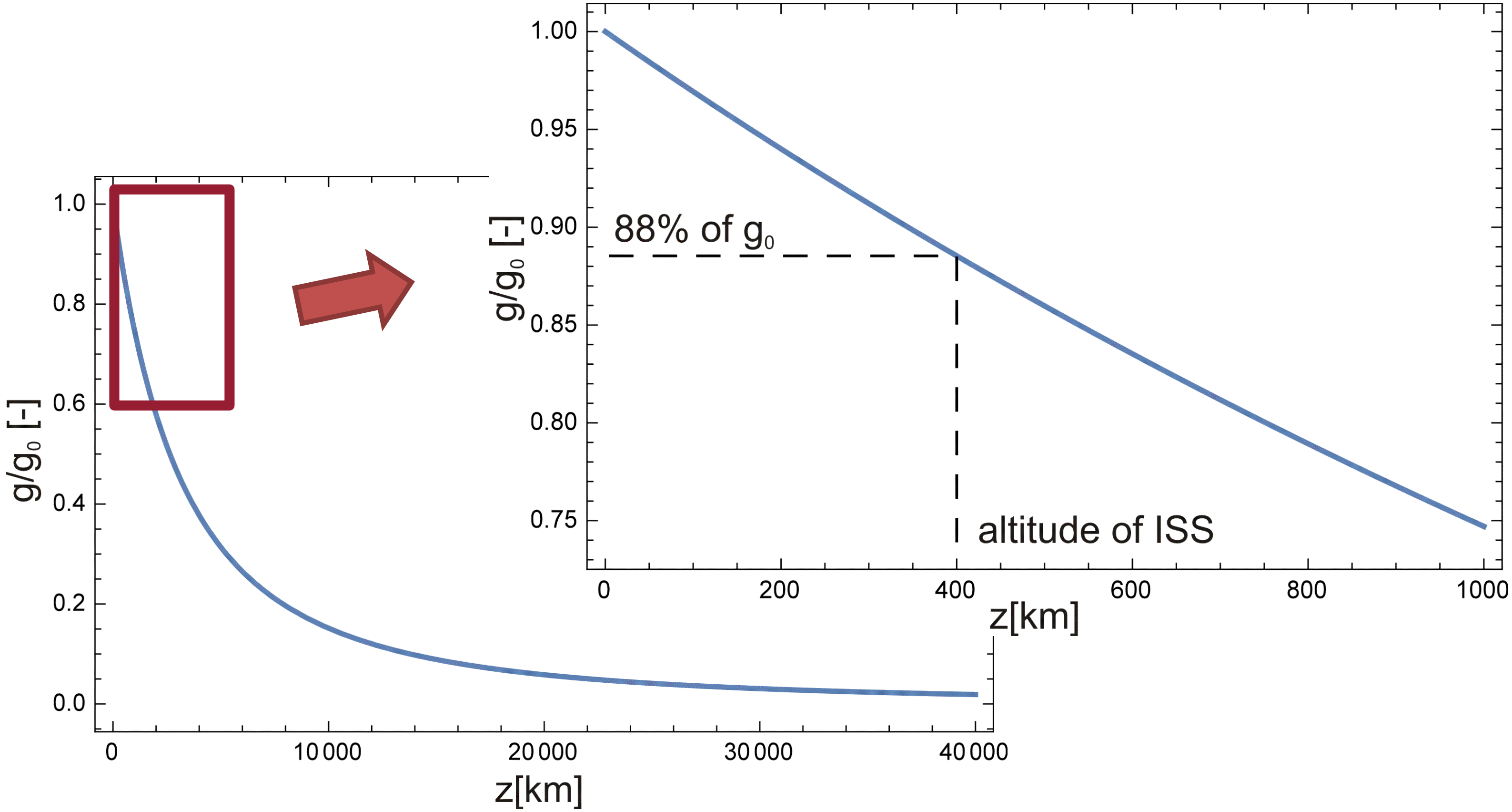

1. The two body problem

Motion in inertial frame

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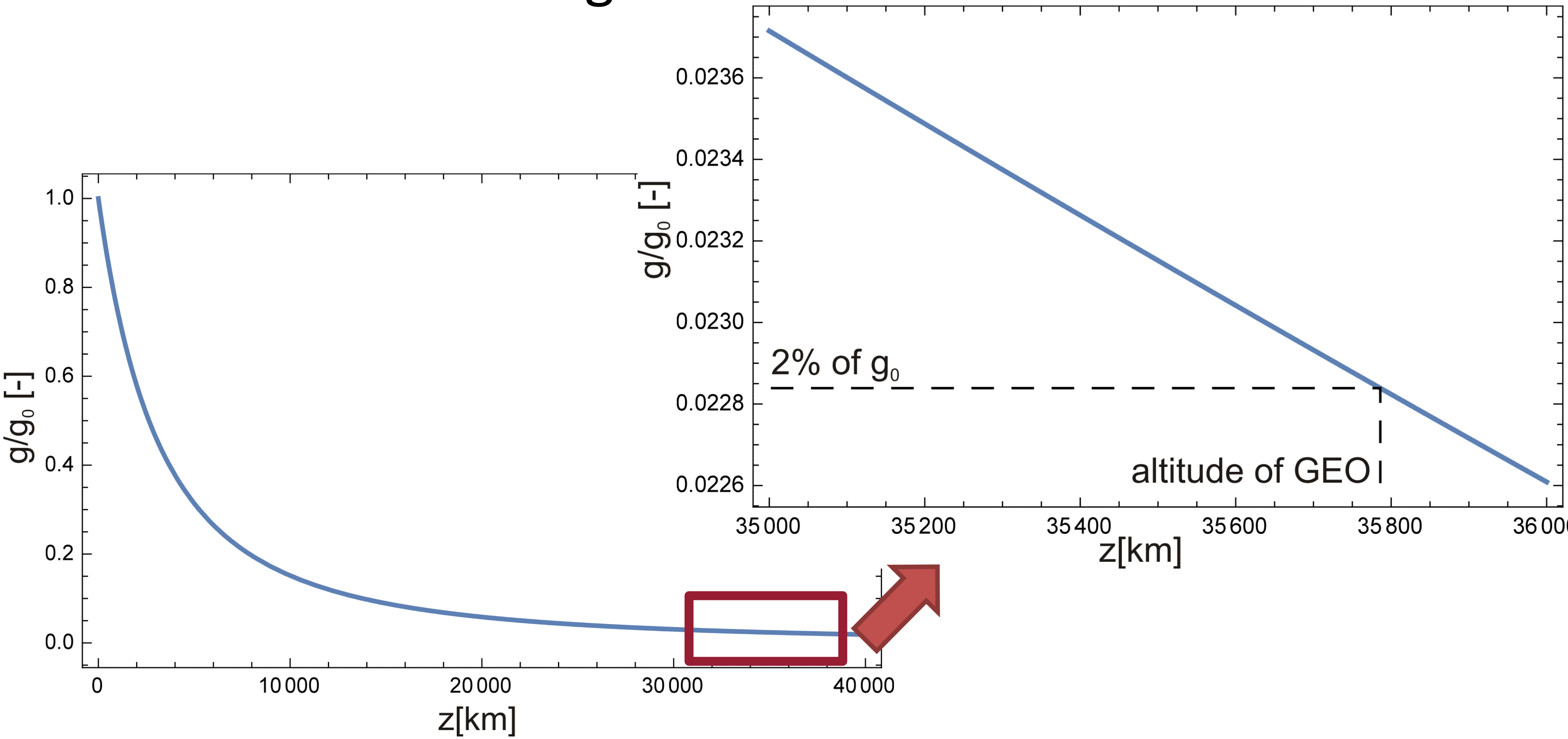
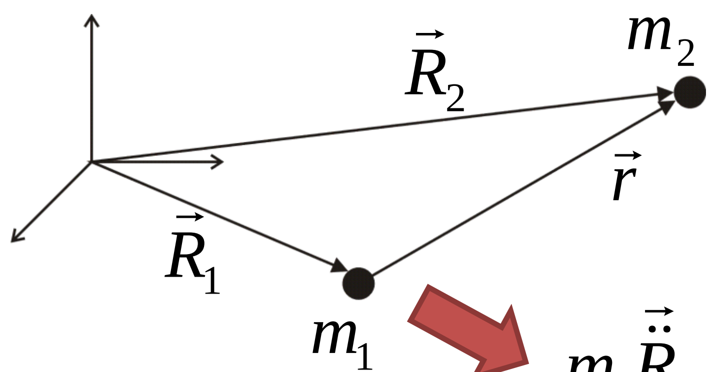


1. The two body problem

Motion in inertial frame

- Two body problem can by defined by:
  - Newton’s law of gravitation $\Rightarrow$ $\vec{F}_{21} = -\vec{F}_{12} = -G\frac{m_1 m_2}{r^3}\vec{r}$
  - Newton's laws of motion

inertial frame of reference



$$m_2\ddot{\vec{R}}_2 = \vec{F}_{21}$$

$$m_1\ddot{\vec{R}}_1 = \vec{F}_{12}$$

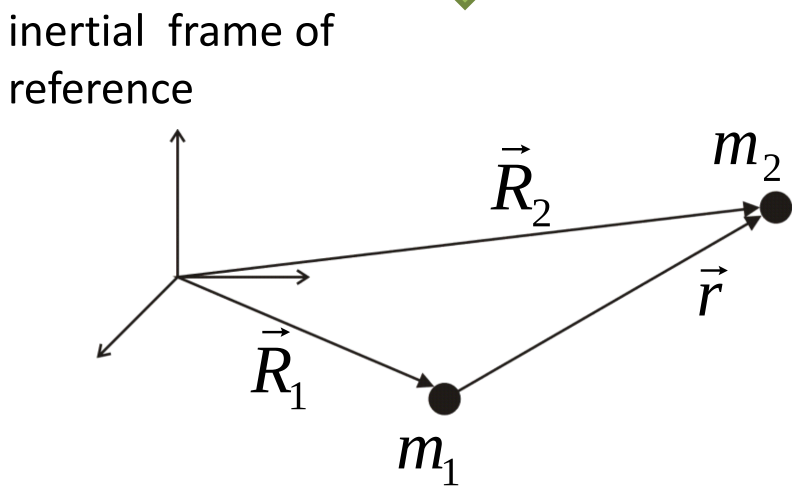
1. The two body problem

Motion in inertial frame

- Two body problem can by defined by:
  - Newton's law of gravitation
  - Newton's laws of motion

$$\vec{F}_{21} = -\vec{F}_{12} = -G\frac{m_1 m_2}{r^3}\vec{r}$$

$$m_1\ddot{\vec{R}}_1 = \vec{F}_{12} \qquad m_2\ddot{\vec{R}}_2 = \vec{F}_{21}$$



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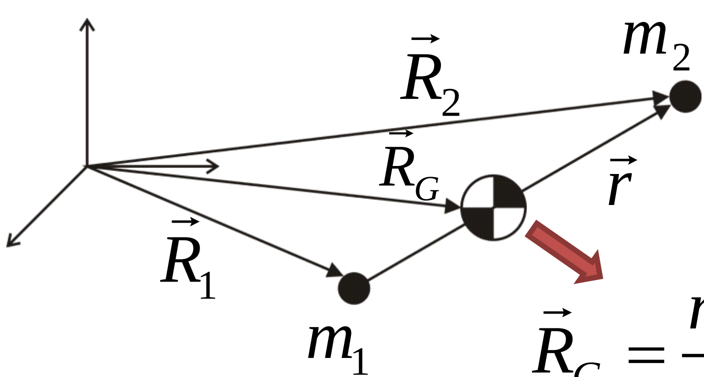
$$m_1\ddot{\vec{R}}_1 = \vec{F}_{12} \qquad m_2\ddot{\vec{R}}_2 = \vec{F}_{21}$$

inertial frame of reference



$$\vec{R}_G = \frac{m_1\vec{R}_1 + m_2\vec{R}_2}{m_1 + m_2}$$

center of mass

1. The two body problem

Motion in inertial frame

- Two body problem can by defined by:
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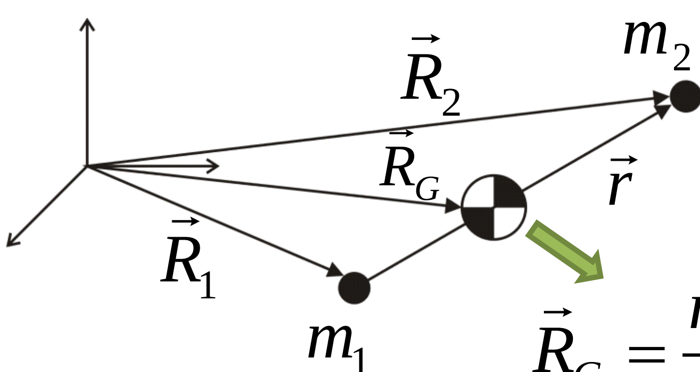
inertial frame of reference



2 x time derivative

$$\ddot{\vec{R}}_G = \frac{m_1\ddot{\vec{R}}_1 + m_2\ddot{\vec{R}}_2}{m_1 + m_2}$$

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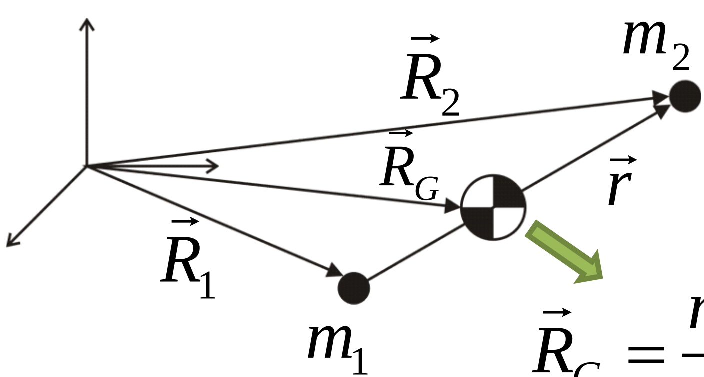
inertial frame of reference



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center of mass

2 x time derivative

$$\ddot{\vec{R}}_G = \frac{m_1\ddot{\vec{R}}_1 + m_2\ddot{\vec{R}}_2}{m_1 + m_2}$$

$$\ddot{\vec{R}}_G = \vec{0}$$

center of mass is:
- motionless
- or motion is in straight line with constant velocity

1. The two body problem

Motion in inertial frame

- Two body problem can by defined by:
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  - Newton's laws of motion

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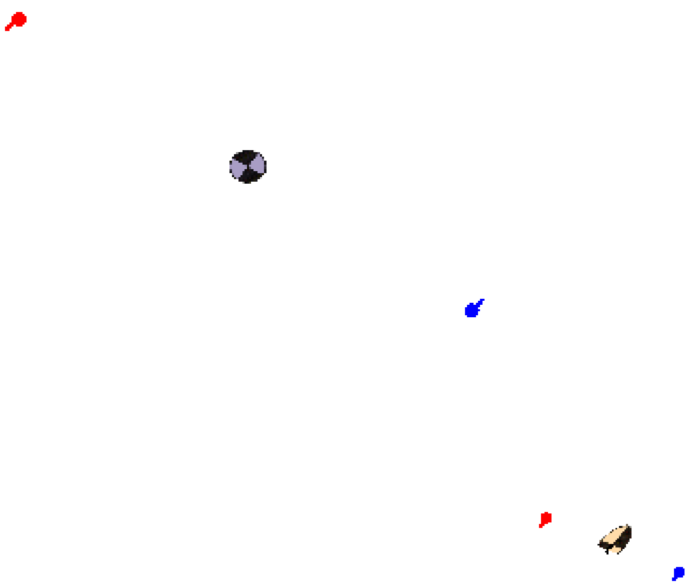

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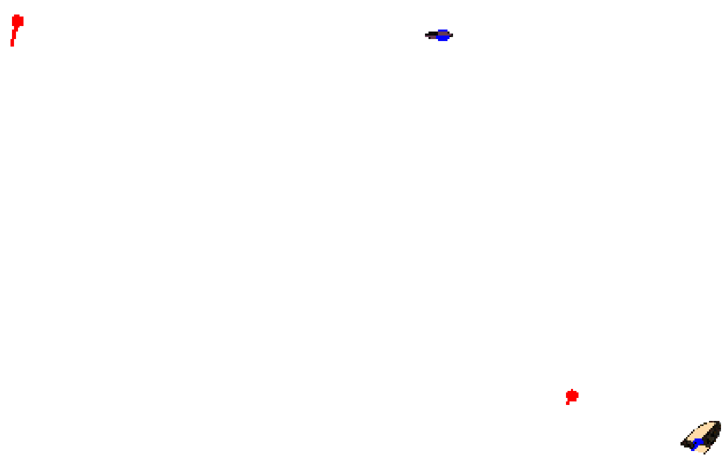

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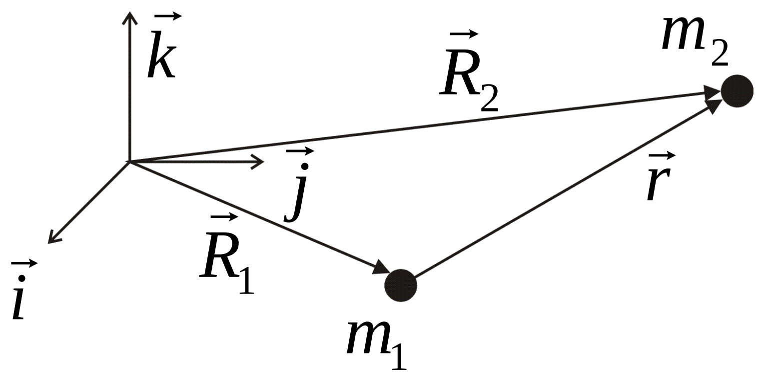
$$m_1\ddot{\vec{R}}_1 = \vec{F}_{12} \qquad m_2\ddot{\vec{R}}_2 = \vec{F}_{21}$$

$$m_1\ddot{\vec{R}}_1 = G\frac{m_1 m_2}{r^3}\vec{r} \qquad m_2\ddot{\vec{R}}_2 = -G\frac{m_1 m_2}{r^3}\vec{r}$$

inertial frame of reference



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Motion in inertial frame

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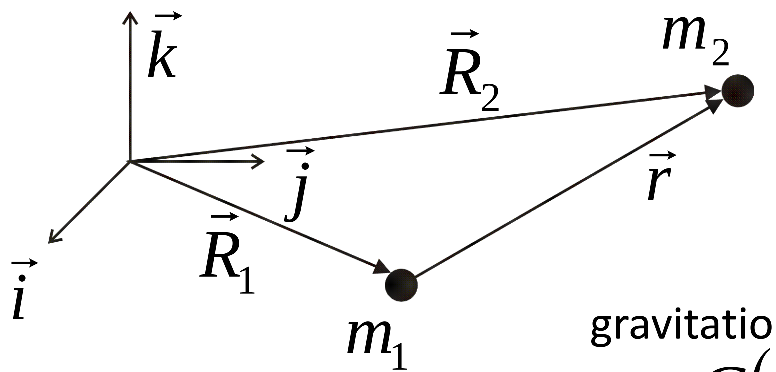
$$m_1\ddot{\vec{R}}_1 = \vec{F}_{12} \qquad m_2\ddot{\vec{R}}_2 = \vec{F}_{21}$$

$$m_1\ddot{\vec{R}}_1 = G\frac{m_1 m_2}{r^3}\vec{r} \qquad m_2\ddot{\vec{R}}_2 = -G\frac{m_1 m_2}{r^3}\vec{r}$$

inertial frame of reference



modification of equations

gravitational parameter

$$\mu = G(m_1 + m_2)$$

if: $m_1 >> m_2$

$$\mu = Gm_1$$

$$\ddot{\vec{r}} + \frac{\mu}{r^3}\vec{r} = 0$$

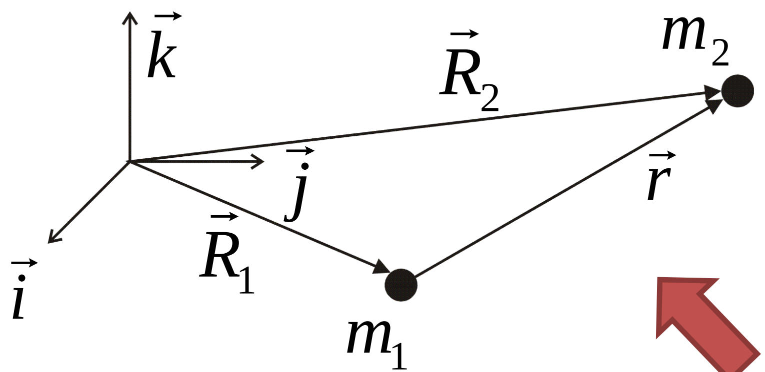
1. The two body problem

Relative motion

- Two body problem can by defined by:
  - Newton's law of gravitation
  - Newton's laws of motion

$$\ddot{\vec{r}} + \frac{\mu}{r^3}\vec{r} = 0$$

inertial frame of reference



vector $\vec{r}$ defined in inertial frame of reference expressed in coord. system $\vec{i}\,\vec{j}\,\vec{k}$

$$\vec{r} = (x_2 - x_1)\vec{i} + (y_2 - y_1)\vec{j} + (z_2 - z_1)\vec{k}$$

1. The two body problem

Relative motion

- Two body problem can by defined by:
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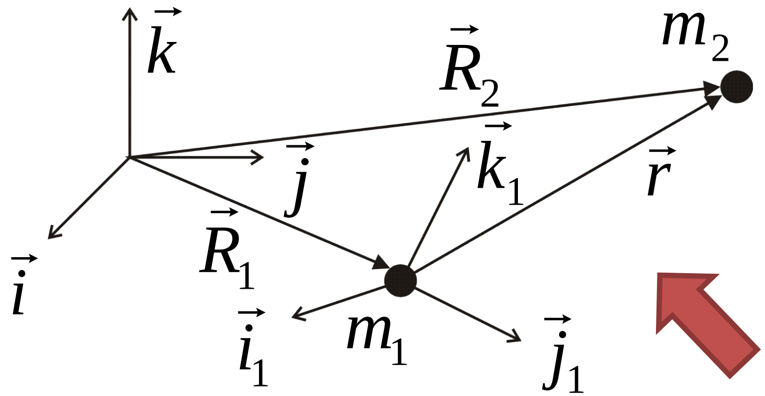
$$\ddot{\vec{r}} + \frac{\mu}{r^3}\vec{r} = 0$$

inertial frame of reference



vector $\vec{r}$ can be expressed in coord. system $\vec{i}_1\vec{j}_1\vec{k}_1$, that rotates about inertial coord. system with instant angular velocity $\vec{\omega}$ and instant angular acceleration $\vec{\alpha}$

$$\vec{r} = (x_{21})\vec{i}_1 + (y_{21})\vec{j}_1 + (z_{21})\vec{k}_1$$

1. The two body problem

Relative motion

- Two body problem can by defined by:
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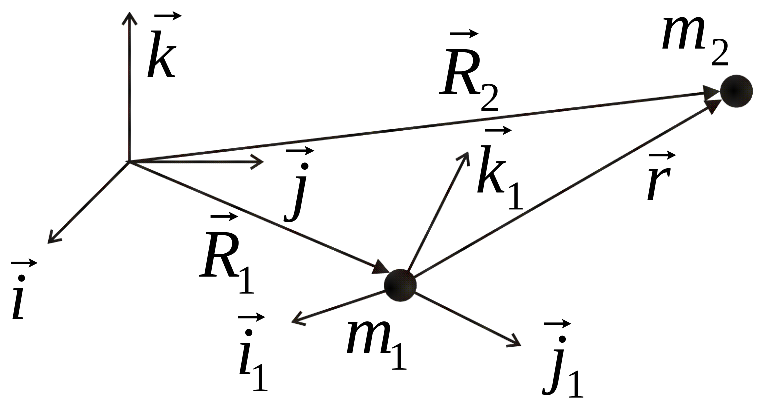
$$\ddot{\vec{r}} + \frac{\mu}{r^3}\vec{r} = 0$$

inertial frame of reference



$$\ddot{\vec{r}} = \ddot{\vec{r}}_{rel} + \vec{\alpha}\times\vec{r} + \vec{\omega}\times\left(\vec{\omega}\times\vec{r}\right) + 2\vec{\omega}\times\vec{v}_{rel}$$

2 x time derivative in inertial frame of reference

$$\vec{r} = (x_{21})\vec{i}_1 + (y_{21})\vec{j}_1 + (z_{21})\vec{k}_1$$

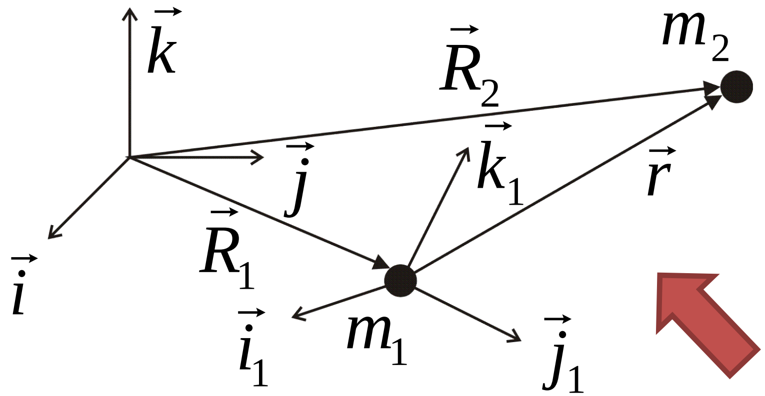
1. The two body problem

Relative motion

- Two body problem can by defined by:
  - Newton’s law of gravitation
  - Newton's laws of motion

$$\ddot{\vec{r}} + \frac{\mu}{r^3}\vec{r} = 0$$

inertial frame of reference



$$\ddot{\vec{r}} = \ddot{\vec{r}}_{rel}$$

if $\vec{i}_1\vec{j}_1\vec{k}_1$ is not rotating coord. system

$$\ddot{\vec{r}} = \ddot{\vec{r}}_{rel} + \vec{\alpha}\times\vec{r} + \vec{\omega}\times\left(\vec{\omega}\times\vec{r}\right) + 2\vec{\omega}\times\vec{v}_{rel}$$

2 x time derivative in inertial frame of reference

$$\vec{r} = (x_{21})\vec{i}_1 + (y_{21})\vec{j}_1 + (z_{21})\vec{k}_1$$

1. The two body problem

Relative motion

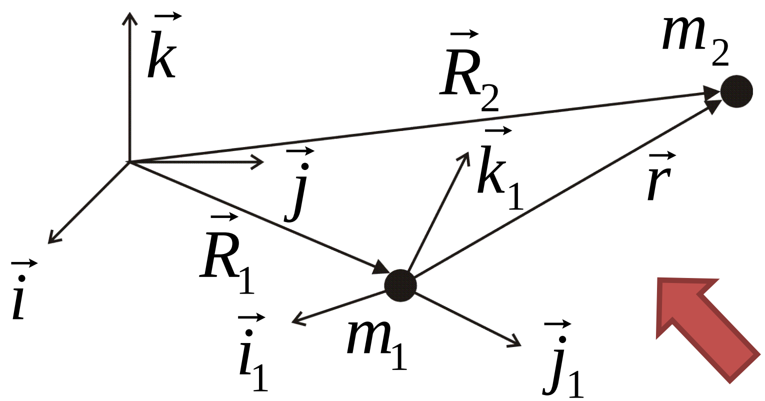
- Two body problem can by defined by:
  - Newton's law of gravitation
  - Newton's laws of motion

$$\ddot{\vec{r}} + \frac{\mu}{r^3}\vec{r} = 0$$

$$\ddot{\vec{r}} = \ddot{\vec{r}}_{rel}$$

relative acceleration of moving (non-rotating) frame of reference in coord. components

inertial frame of reference



$$\ddot{x}_{21} = -\frac{\mu}{r^3}x_{21}$$

$$\ddot{y}_{21} = -\frac{\mu}{r^3}y_{21}$$

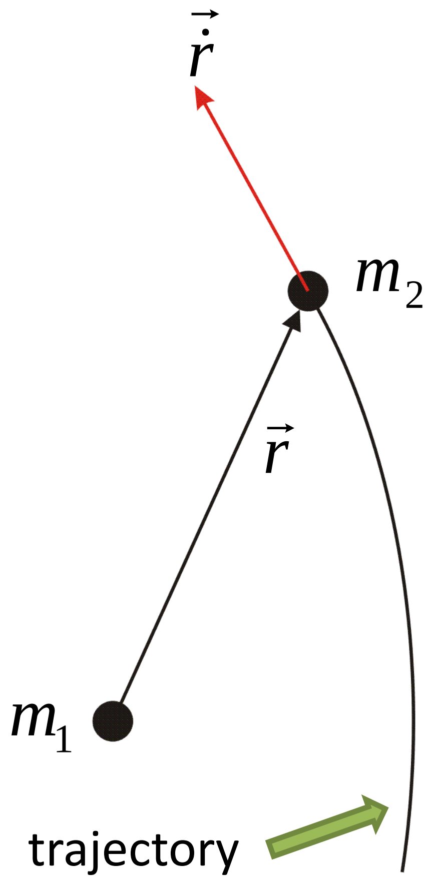
$$\ddot{z}_{21} = -\frac{\mu}{r^3}z_{21}$$

+ 6 initial conditions

1. The two body problem

Angular momentum

- relative angular momentum of body m_2 per unit mass



$$\vec{h} = \frac{1}{m_2}\left(\vec{r} \times m_2 \dot{\vec{r}}\right) = \vec{r} \times \dot{\vec{r}}$$

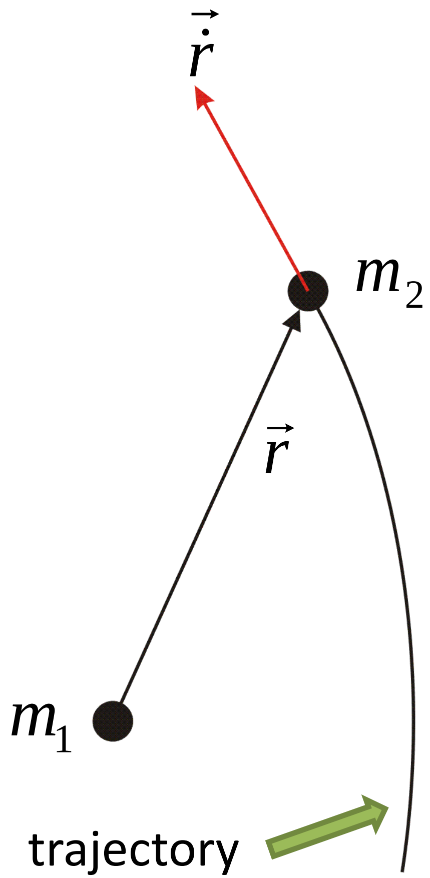
1 x time derivative

$$\frac{d\vec{h}}{dt} = \dot{\vec{r}} \times \dot{\vec{r}} + \vec{r} \times \ddot{\vec{r}} = \vec{r} \times \ddot{\vec{r}}$$

1. The two body problem

Angular momentum

- relative angular momentum of body m_2 per unit mass



$$\vec{h} = \frac{1}{m_2}\left(\vec{r} \times m_2 \dot{\vec{r}}\right) = \vec{r} \times \dot{\vec{r}}$$

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$$\ddot{\vec{r}} + \frac{\mu}{r^3}\vec{r} = 0$$

1. The two body problem

Angular momentum

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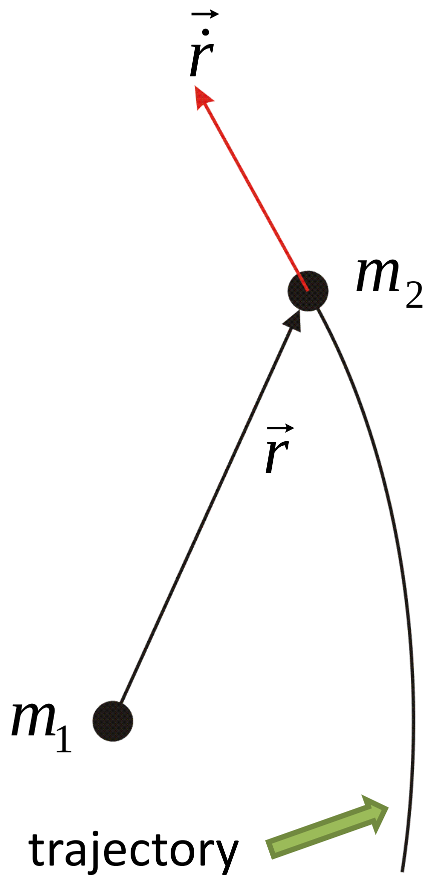


$$\vec{h} = \frac{1}{m_2}\left(\vec{r} \times m_2 \vec{\dot{r}}\right) = \vec{r} \times \vec{\dot{r}}$$

1 x time derivative

$$\frac{d\vec{h}}{dt} = \vec{\dot{r}} \times \vec{\dot{r}} + \vec{r} \times \vec{\ddot{r}} = \vec{r} \times \vec{\ddot{r}}$$

$$\vec{\ddot{r}} + \frac{\mu}{r^3}\vec{r} = 0$$

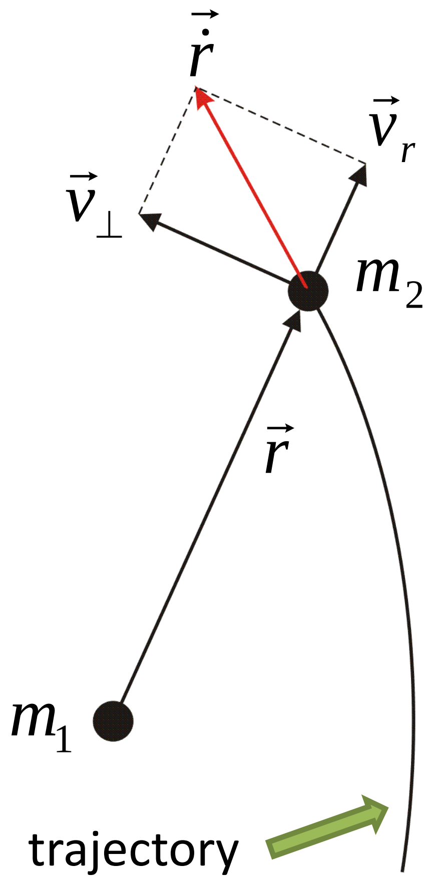
angular momentum is conserved

$$\frac{d\vec{h}}{dt} = \vec{0}$$

1. The two body problem

Angular momentum

- relative angular momentum of body m_2 per unit mass



velocity vector can be expressed as

$$\dot{\vec{r}} = \vec{v} = \vec{v}_\perp + \vec{v}_r$$

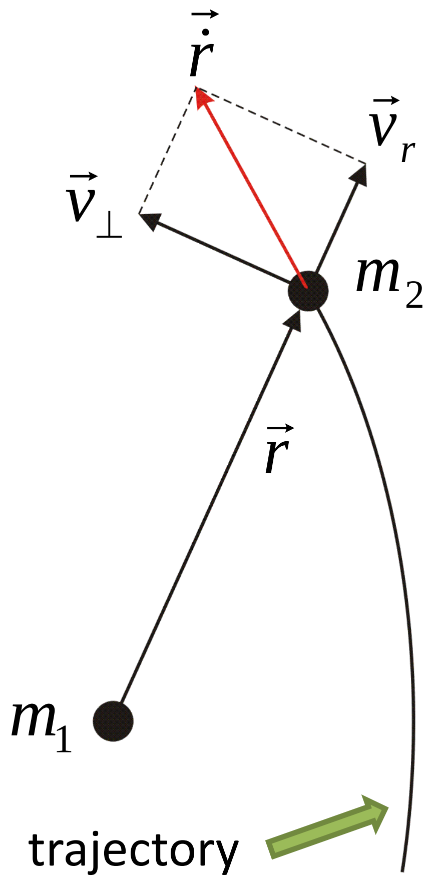
angular momentum can be expressed as

$$\vec{h} = \vec{r} \times \dot{\vec{r}} = \vec{r} \times \vec{v}_\perp + \vec{r} \times \vec{v}_r = \vec{r} \times \vec{v}_\perp$$

1. The two body problem

Angular momentum

- relative angular momentum of body m_2 per unit mass



velocity vector can be expressed as

$$\vec{\dot{r}} = \vec{v} = \vec{v}_\perp + \vec{v}_r$$

angular momentum can be expressed as

$$\vec{h} = \vec{r} \times \vec{\dot{r}} = \vec{r} \times \vec{v}_\perp + \vec{r} \times \vec{v}_r = \vec{r} \times \vec{v}_\perp$$

$$\vec{h} = \vec{r} \times \vec{v}_\perp = r v_\perp \vec{k}_1 = h \vec{k}_1$$

$$h = r v_\perp$$

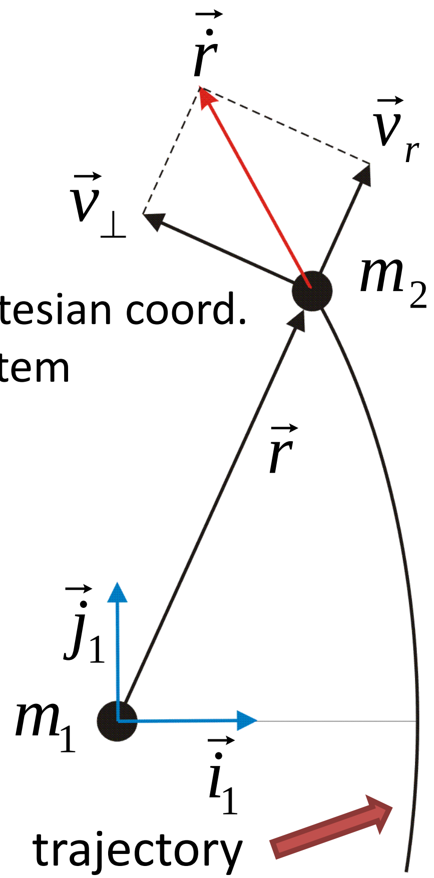
$\vec{k}_1$
- unit vector
- time invariant

h
- magnitude of angular momentum
- time invariant

1. The two body problem

Angular momentum

- relative angular momentum of body m_2 per unit mass



- Cartesian coord. system

$$\vec{h} = \vec{r} \times \vec{v}_\perp = r v_\perp \vec{k}_1 = h \vec{k}_1$$

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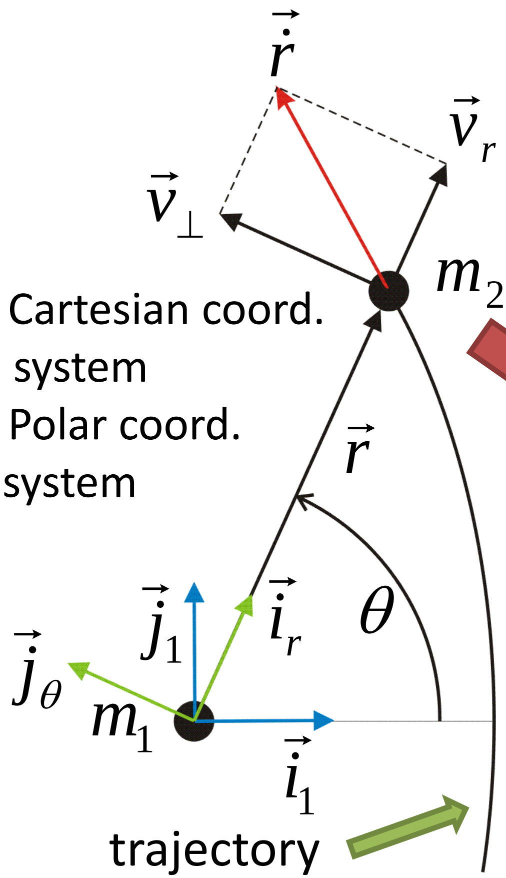

- Cartesian coord. system
- Polar coord. system

$$v_\perp = \frac{d\theta}{dt} r = \dot{\theta}\, r$$

$$\vec{h} = \vec{r} \times \vec{v}_\perp = r v_\perp \vec{k}_1 = h \vec{k}_1$$

$\vec{k}_1$
- unit vector
- time invariant

h
- magnitude of angular momentum
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1. The two body problem

Angular momentum

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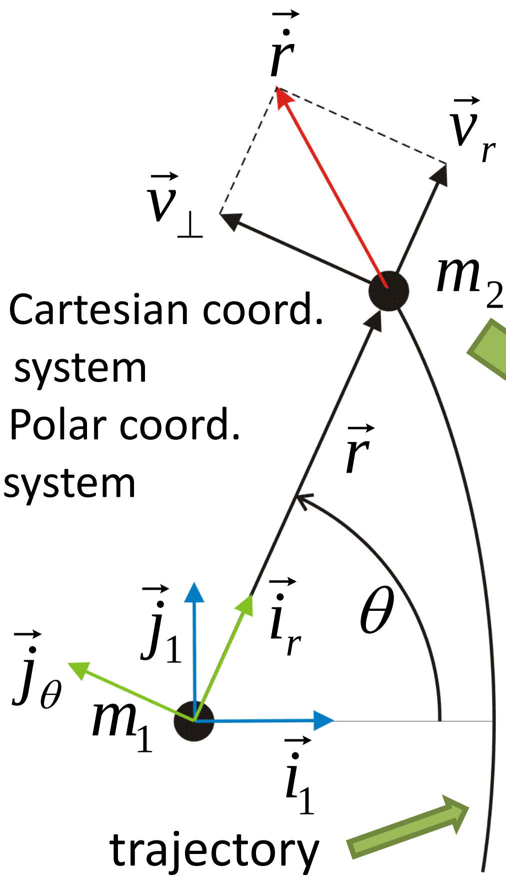


- Cartesian coord. system
- Polar coord. system

$$\vec{h} = r\left(\dot{\theta}\, r\right)\vec{k}_1 = \dot{\theta}\, r^2 \vec{k}_1 = h\vec{k}_1$$

$$v_\perp = \frac{d\theta}{dt} r = \dot{\theta}\, r$$

$$\vec{h} = \vec{r} \times \vec{v}_\perp = r v_\perp \vec{k}_1 = h\vec{k}_1$$

$\vec{k}_1$
- unit vector
- time invariant

h
- magnitude of angular momentum
- time invariant

1. The two body problem

Solution of problem

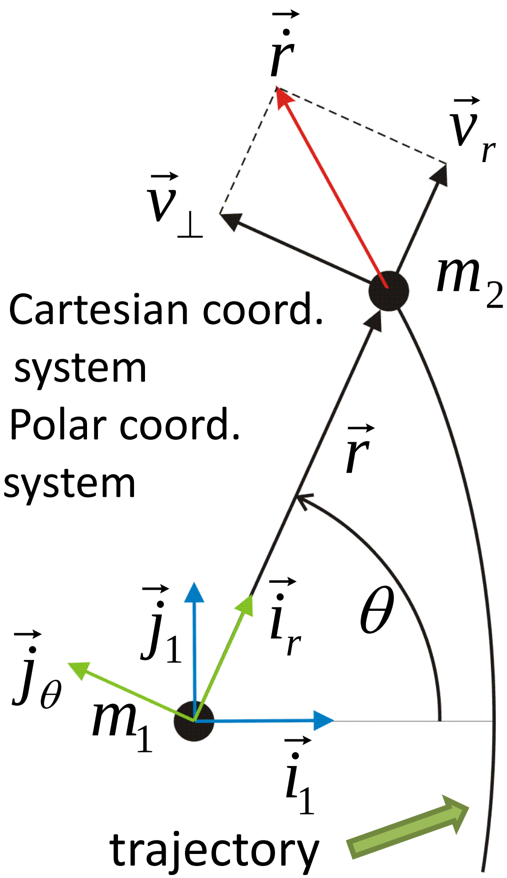
- Equation of orbit

$$\ddot{\vec{r}} + \frac{\mu}{r^3}\vec{r} = 0$$

cross product with $\vec{h}$

$$\ddot{\vec{r}} \times \vec{h} = -\frac{\mu}{r^3}\vec{r} \times \vec{h}$$

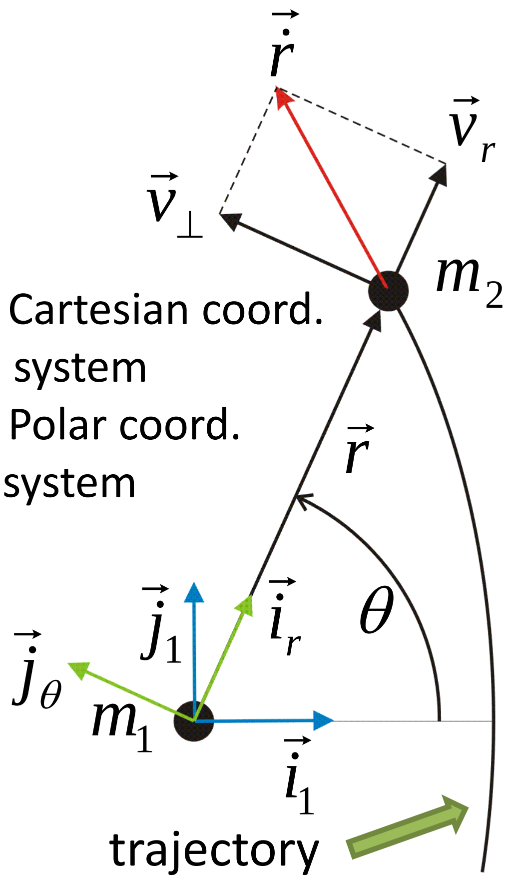
- Cartesian coord. system
- Polar coord. system



1. The two body problem

Solution of problem

- Equation of orbit



- Cartesian coord. system
- Polar coord. system

$$\vec{r} = r\vec{i}_r \qquad \vec{h} = \dot{\theta}\, r^2 \vec{k}_1$$

$\vec{r}$ and $\vec{h}$ expressed by polar coordinates

$$\ddot{\vec{r}} + \frac{\mu}{r^3}\vec{r} = 0$$

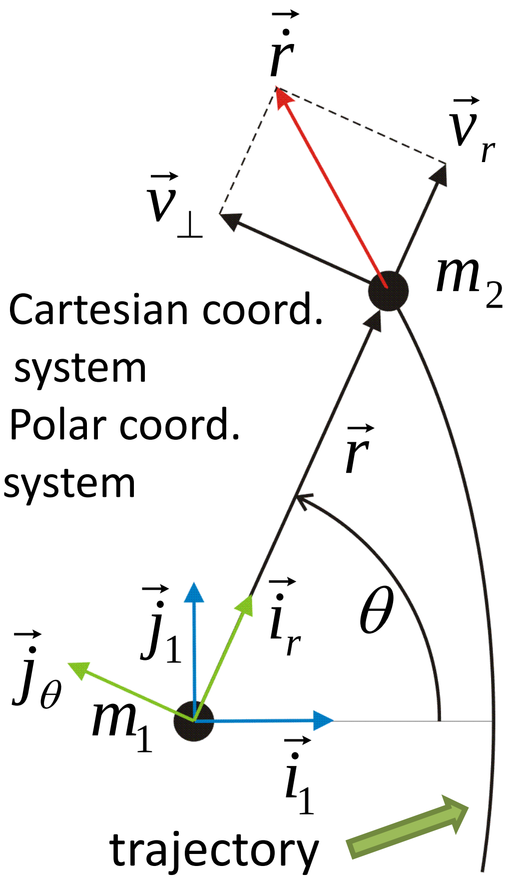
cross product with $\vec{h}$

$$\ddot{\vec{r}} \times \vec{h} = -\frac{\mu}{r^3}\vec{r} \times \vec{h}$$

1. The two body problem

Solution of problem

- Equation of orbit



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$$\ddot{\vec{r}} \times \vec{h} = \mu\dot{\theta}\, \vec{j}_\theta$$

1. The two body problem

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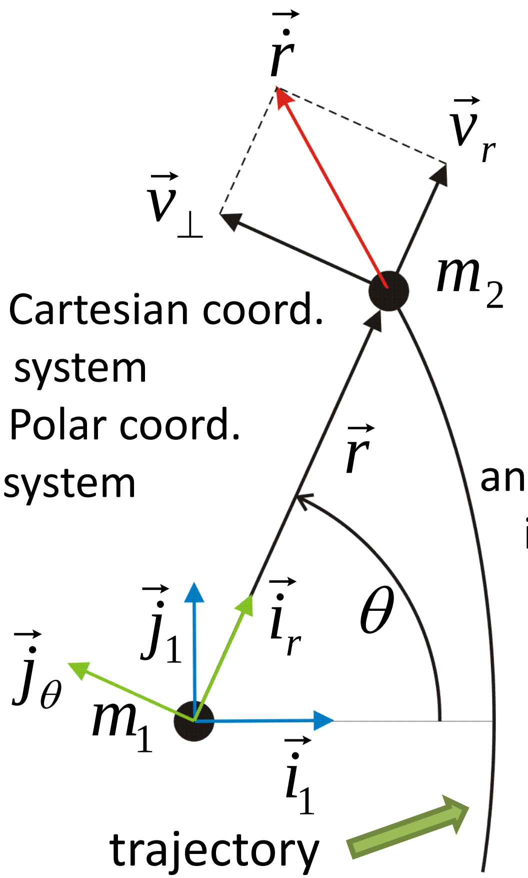
angular momentum is const. vector

$$\frac{d\vec{h}}{dt} = \vec{0}$$



- Cartesian coord. system
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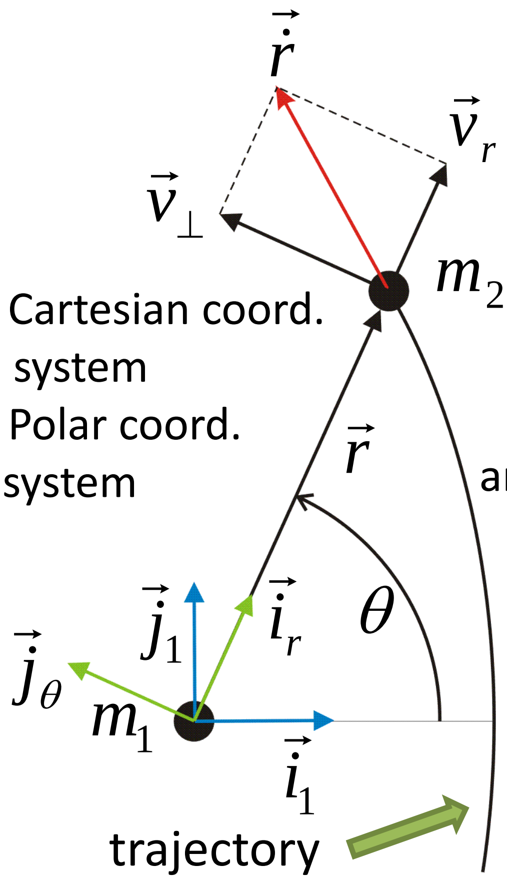
1. The two body problem

Solution of problem

- Equation of orbit



- Cartesian coord. system
- Polar coord. system

$$\vec{\ddot{r}} + \frac{\mu}{r^3}\vec{r} = 0$$

cross product with $\vec{h}$

$\vec{r}$ and $\vec{h}$ expressed by polar coordinates

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$$\vec{\ddot{r}} \times \vec{h} = \mu\dot{\theta}\,\vec{j}_\theta$$

angular momentum is const. vector

$$\frac{d\vec{h}}{dt} = \vec{0}$$

$$\frac{d}{dt}\left(\vec{\dot{r}} \times \vec{h}\right) = \mu\frac{d\theta}{dt}\,\vec{j}_\theta$$

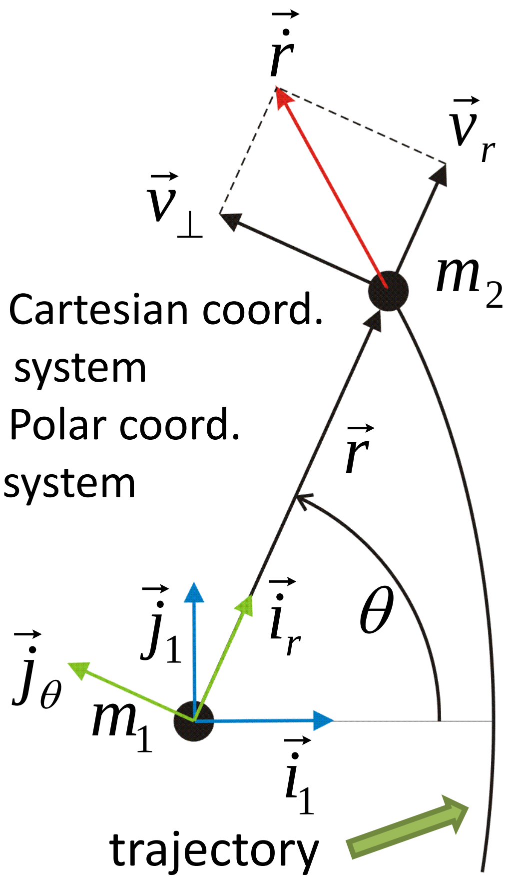
1. The two body problem

Solution of problem

- Equation of orbit

$$\frac{d}{dt}\left(\dot{\vec{r}} \times \vec{h}\right) = \mu \frac{d\theta}{dt} \vec{j}_\theta$$

- Cartesian coord. system
- Polar coord. system



- Unit vectors of polar coord. system are not constant vectors

$$\vec{i}_r = \cos\theta\,\vec{i}_1 + \sin\theta\,\vec{j}_1$$

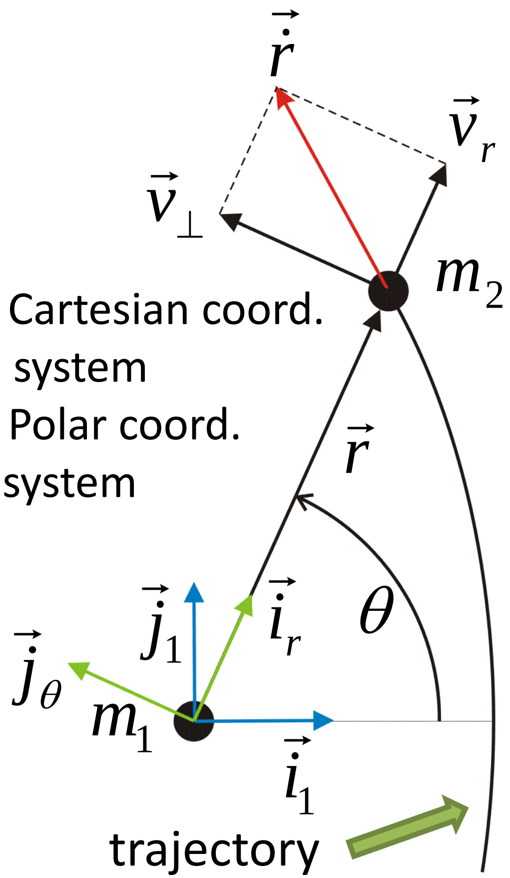
$$\vec{j}_\theta = -\sin\theta\,\vec{i}_1 + \cos\theta\,\vec{j}_1$$

1. The two body problem

Solution of problem

- Equation of orbit

$$\frac{d}{dt}\left(\dot{\vec{r}} \times \vec{h}\right) = \mu \frac{d\theta}{dt}\,\vec{j}_\theta$$



- Cartesian coord. system
- Polar coord. system

- Unit vectors of polar coord. system are not constant vectors

$$\frac{d\vec{i}_r}{d\theta} = -\sin\theta\,\vec{i}_1 + \cos\theta\,\vec{j}_1$$

$$\vec{i}_r = \cos\theta\,\vec{i}_1 + \sin\theta\,\vec{j}_1$$

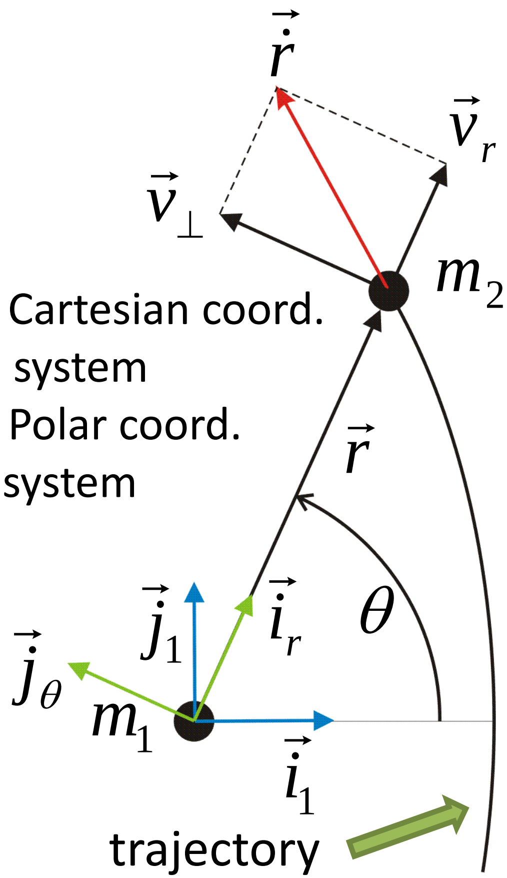
$$\vec{j}_\theta = -\sin\theta\,\vec{i}_1 + \cos\theta\,\vec{j}_1$$

1. The two body problem

Solution of problem

- Equation of orbit

$$\frac{d}{dt}\left(\dot{\vec{r}} \times \vec{h}\right) = \mu \frac{d\theta}{dt} \vec{j}_\theta$$



- Cartesian coord. system
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- Unit vectors of polar coord. system are not constant vectors

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$$\vec{j}_\theta = -\sin\theta\, \vec{i}_1 + \cos\theta\, \vec{j}_1$$

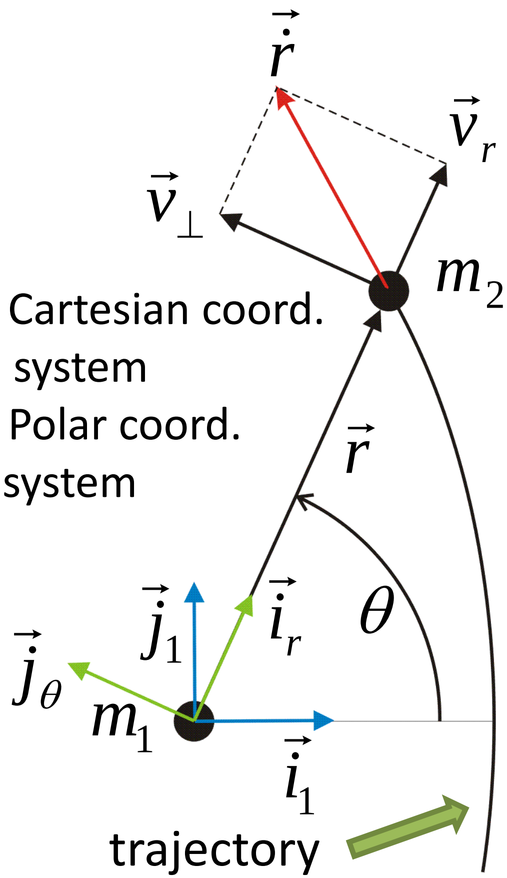
$$\frac{d\vec{i}_r}{d\theta} = -\sin\theta\, \vec{i}_1 + \cos\theta\, \vec{j}_1$$

$$\frac{d\vec{i}_r}{d\theta} = \vec{j}_\theta$$

1. The two body problem

Solution of problem

- Equation of orbit



- Cartesian coord. system
- Polar coord. system

- μ is scalar – time invariant parameter

$$\frac{d}{dt}\left(\dot{\vec{r}} \times \vec{h}\right) = \mu \frac{d}{dt}\left(\vec{i}_r\right)$$

- Unit vectors of polar coord. system are not constant vectors

$$\vec{i}_r = \cos\theta\, \vec{i}_1 + \sin\theta\, \vec{j}_1$$

$$\vec{j}_\theta = -\sin\theta\, \vec{i}_1 + \cos\theta\, \vec{j}_1$$

$$\frac{d\vec{i}_r}{d\theta} = -\sin\theta\, \vec{i}_1 + \cos\theta\, \vec{j}_1$$

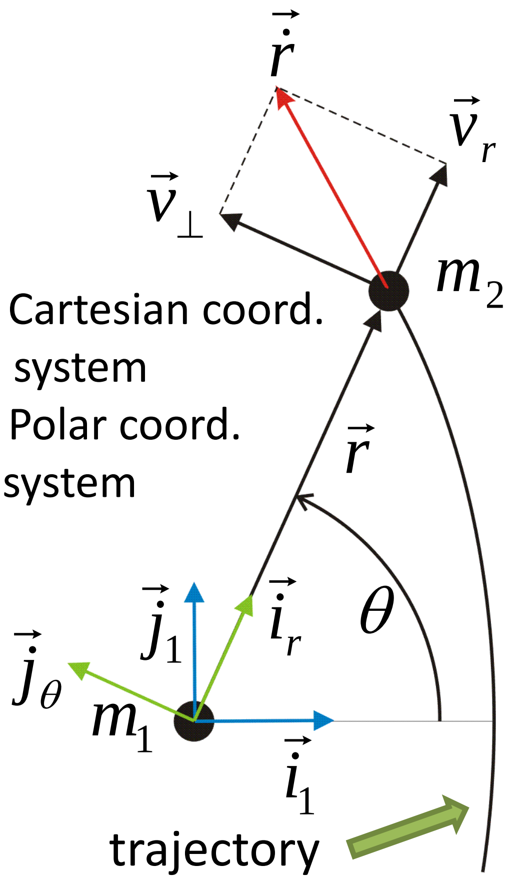
$$\frac{d\vec{i}_r}{d\theta} = \vec{j}_\theta$$

$$\frac{d}{dt}\left(\dot{\vec{r}} \times \vec{h}\right) = \mu \frac{d\theta}{dt} \vec{j}_\theta$$

1. The two body problem

Solution of problem

- Equation of orbit



- Cartesian coord. system
- Polar coord. system

$$\frac{d}{dt}\left(\dot{\vec r}\times\vec h\right)=\mu\frac{d\theta}{dt}\,\vec j_\theta$$

- μ is scalar – time invariant parameter

$$\frac{d}{dt}\left(\dot{\vec r}\times\vec h\right)=\mu\frac{d}{dt}\left(\vec i_r\right)$$

- $\vec e$ is integration constant, i.e. const. vector

$$\left(\dot{\vec r}\times\vec h\right)=\mu\left(\vec i_r+\vec e\right)$$

1. The two body problem

Solution of problem

- Equation of orbit

$$\frac{d}{dt}\left(\dot{\vec{r}}\times\vec{h}\right)=\mu\frac{d\theta}{dt}\vec{j}_\theta$$

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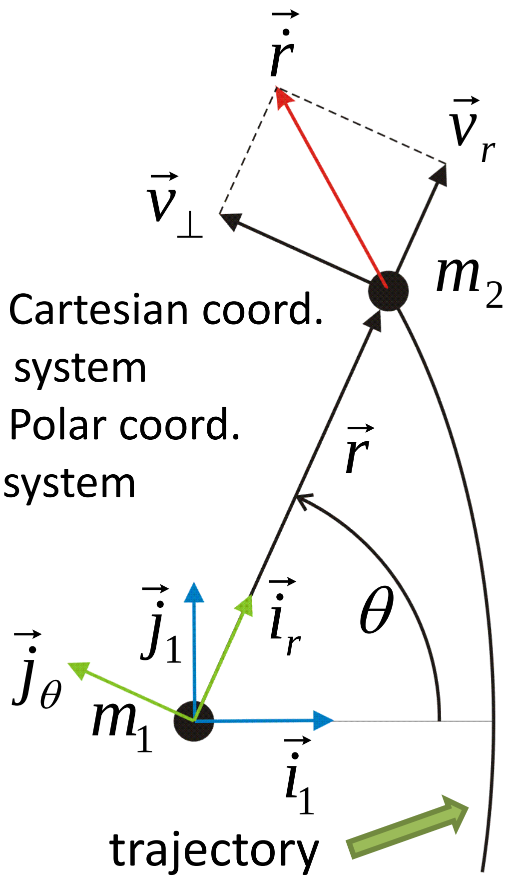
$$\frac{d}{dt}\left(\dot{\vec{r}}\times\vec{h}\right)=\mu\frac{d}{dt}\left(\vec{i}_r\right)$$

- $\vec{e}$ is integration constant, i.e. const. vector

$$\left(\dot{\vec{r}}\times\vec{h}\right)=\mu\left(\vec{i}_r+\vec{e}\right)$$

- dot product with $\vec{r}$

$$\vec{r}\cdot\left(\dot{\vec{r}}\times\vec{h}\right)=\mu\,\vec{r}\cdot\left(\vec{i}_r+\vec{e}\right)$$

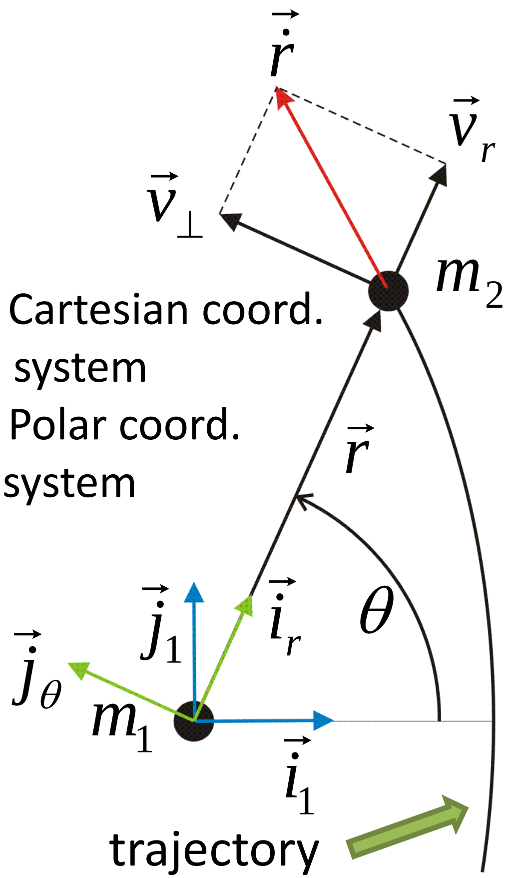


- Cartesian coord. system
- Polar coord. system

1. The two body problem

Solution of problem

- Equation of orbit



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$$\left(\vec{\dot{r}}\times\vec{h}\right)=\mu\left(\vec{i}_r+\vec{e}\right)$$

- dot product with $\vec{r}$

$$\vec{r}\cdot\left(\vec{\dot{r}}\times\vec{h}\right)=\mu\,\vec{r}\cdot\left(\vec{i}_r+\vec{e}\right)$$

- using $\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)=\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}$

$$\left(\vec{r}\times\vec{\dot{r}}\right)\cdot\vec{h}=\mu\,r\left(1+e\cos\theta\right)$$

1. The two body problem

Solution of problem

- Equation of orbit

$$\frac{d}{dt}\left(\dot{\vec{r}}\times\vec{h}\right)=\mu\frac{d\theta}{dt}\vec{j}_\theta$$

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$$\frac{d}{dt}\left(\dot{\vec{r}}\times\vec{h}\right)=\mu\frac{d}{dt}\left(\vec{i}_r\right)$$

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$$\left(\dot{\vec{r}}\times\vec{h}\right)=\mu\left(\vec{i}_r+\vec{e}\right)$$

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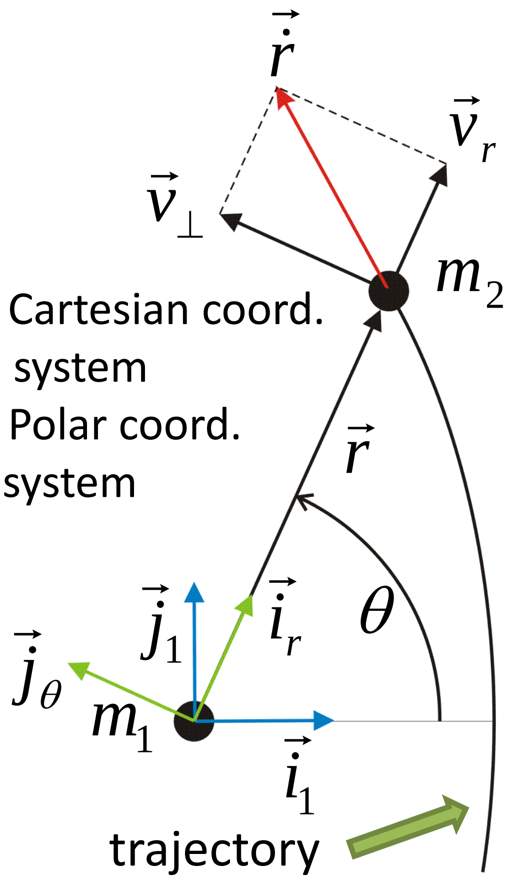
$$\left(\vec{r}\times\dot{\vec{r}}\right)\cdot\vec{h}=\mu\,r\left(1+e\cos\theta\right)$$

$$r=\frac{h^2}{\mu}\frac{1}{\left(1+e\cos\theta\right)}$$

- Scalar equation of orbit $r(\theta)$
- e is eccentricity

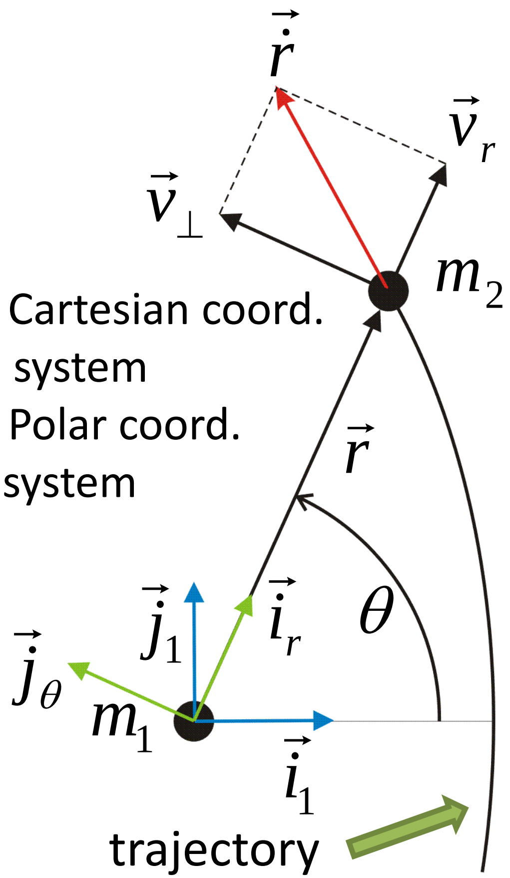


- Cartesian coord. system
- Polar coord. system

1. The two body problem

Solution of problem

- Equation of orbit



- Cartesian coord. system
- Polar coord. system

$$v_\perp = \frac{h}{r} = \frac{\mu}{h}\left(1 + e\cos\theta\right)$$

$$h = r v_\perp$$

$$r = \frac{h^2}{\mu}\frac{1}{\left(1 + e\cos\theta\right)}$$

- Scalar equation of orbit $r(\theta)$
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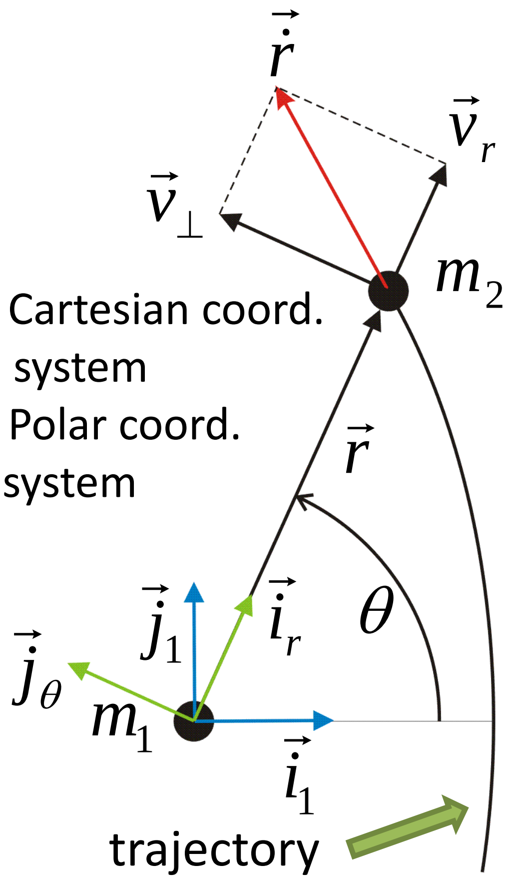
1. The two body problem

Solution of problem

- Equation of orbit



- Cartesian coord. system
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$$v_\perp = \frac{h}{r} = \frac{\mu}{h}(1 + e\cos\theta)$$

$$h = r v_\perp$$

$$v_r = \dot{r} = \frac{dr}{dt} = \frac{\mu}{h} e \sin\theta$$

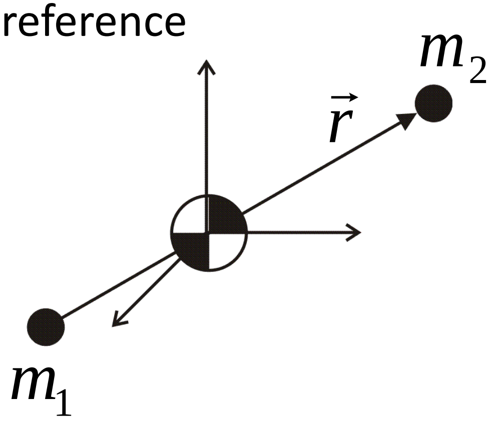
$$r = \frac{h^2}{\mu} \frac{1}{(1 + e\cos\theta)}$$

- Scalar equation of orbit $r(\theta)$
- e is eccentricity

1. The two body problem

Energy law

- energy of system written in inertial frame of reference placed in center of mass $E_{tot} = E_{k1} + E_{k2} + E_p$



1. The two body problem

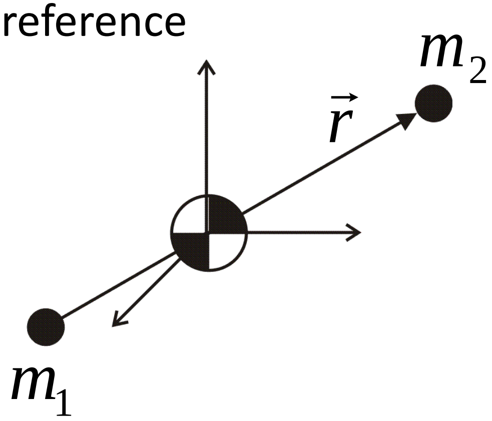
Energy law

- energy of system written in inertial frame of reference placed in center of mass $E_{tot} = E_{k1} + E_{k2} + E_p$

$$E_{tot} = \frac{1}{2} m_1 v_{m1}^2 + \frac{1}{2} m_2 v_{m2}^2 - G \frac{m_1 m_2}{r}$$

expressed by inertial motion

inertial frame of reference



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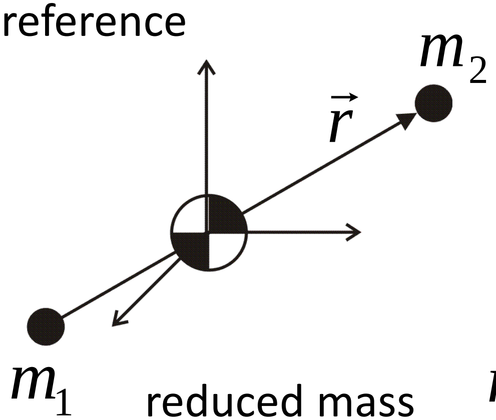
Energy law

- energy of system written in inertial frame of reference placed in center of mass $E_{tot} = E_{k1} + E_{k2} + E_p$

$$E_{tot} = \frac{1}{2} m_1 v_{m1}^2 + \frac{1}{2} m_2 v_{m2}^2 - G\frac{m_1 m_2}{r}$$

expressed by inertial motion

inertial frame of reference



$$E_{tot} = \frac{1}{2}\frac{m_1 m_2}{m_1 + m_2} v^2 - G\frac{m_1 m_2}{r}$$

expressed by relative motion

reduced mass of system $\dfrac{m_1 m_2}{m_1 + m_2}$

1. The two body problem

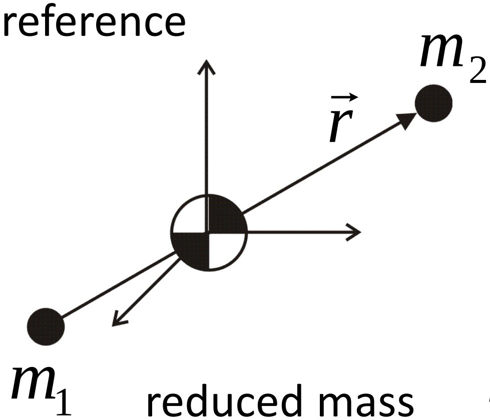
Energy law

- energy of system written in inertial frame of reference placed in center of mass $E_{tot} = E_{k1} + E_{k2} + E_p$

$$E_{tot} = \frac{1}{2} m_1 v_{m1}^2 + \frac{1}{2} m_2 v_{m2}^2 - G\frac{m_1 m_2}{r}$$

expressed by inertial motion

inertial frame of reference



$$E_{tot} = \frac{1}{2}\frac{m_1 m_2}{m_1 + m_2} v^2 - G\frac{m_1 m_2}{r}$$

expressed by relative motion

reduced mass of system $\dfrac{m_1 m_2}{m_1 + m_2}$

$$\varepsilon = \frac{v^2}{2} - \frac{\mu}{r}$$

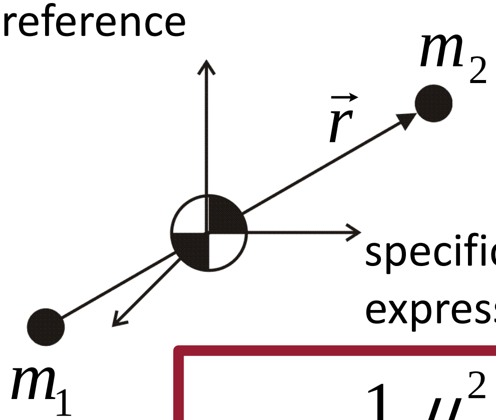
specific orbital energy (total energy per unit reduced mass)
vis viva equation

1. The two body problem

Energy law

- energy of system written in inertial frame of reference placed in center of mass $E_{tot} = E_{k1} + E_{k2} + E_p$

inertial frame of reference



$$E_{tot} = \frac{1}{2} m_1 v_{m1}^2 + \frac{1}{2} m_2 v_{m2}^2 - G\frac{m_1 m_2}{r}$$

expressed by inertial motion

$$E_{tot} = \frac{1}{2}\frac{m_1 m_2}{m_1 + m_2} v^2 - G\frac{m_1 m_2}{r}$$

expressed by relative motion

$$\varepsilon = \frac{v^2}{2} - \frac{\mu}{r}$$

specific orbital energy (total energy per unit reduced mass)

vis viva equation

specific energy expressed by e

$$\varepsilon = -\frac{1}{2}\frac{\mu^2}{h^2}\left(1 - e^2\right)$$

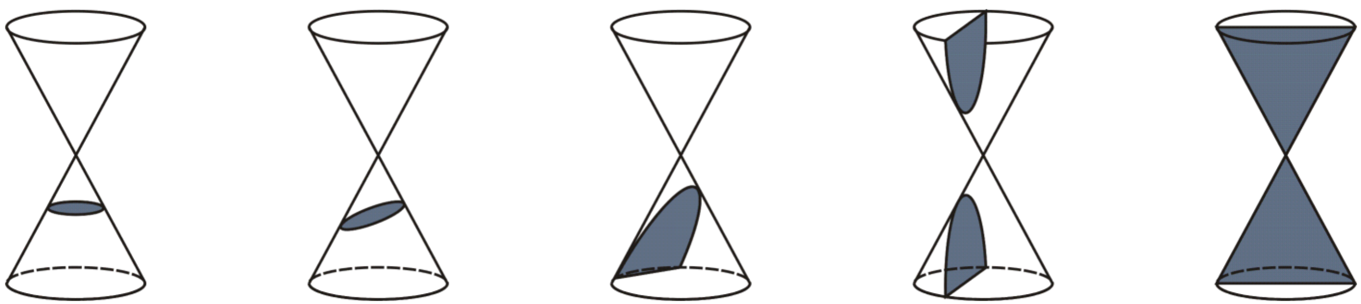
1. The two body problem

Trajectories

- Shape of trajectory depends on eccentricity e
- Equation of orbit is equation of conic sections:
  - circle $e = 0$
  - ellipse $0 < e < 1$
  - parabola $e = 1$
  - hyperbola $e > 1$

- equation of orbit

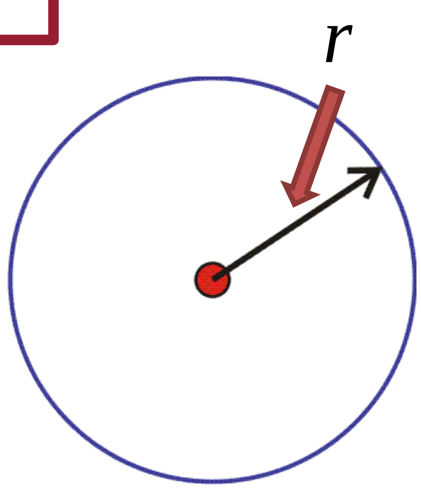
$$r = \frac{h^2}{\mu} \frac{1}{(1 + e\cos\theta)}$$



1. The two body problem

Trajectories

- **Circle**: (bounded trajectory) $e = 0$
  - equation of orbit $$r = \frac{h^2}{\mu}\frac{1}{(1+e\cos\theta)} \Rightarrow r = \frac{h^2}{\mu}$$
  - speed of motion $$v = v_\perp = \frac{\mu}{h}(1+e\cos\theta) \Rightarrow v = \sqrt{\frac{\mu}{r}}$$
  - period $$T = \frac{2\pi r}{\sqrt{\mu / r}} \Rightarrow T = \frac{2\pi}{\sqrt{\mu}} r^{3/2}$$
  - specific energy $$\varepsilon = -\frac{1}{2}\frac{\mu^2}{h^2}(1-e^2) \Rightarrow \varepsilon = -\frac{1}{2}\frac{\mu}{r}$$



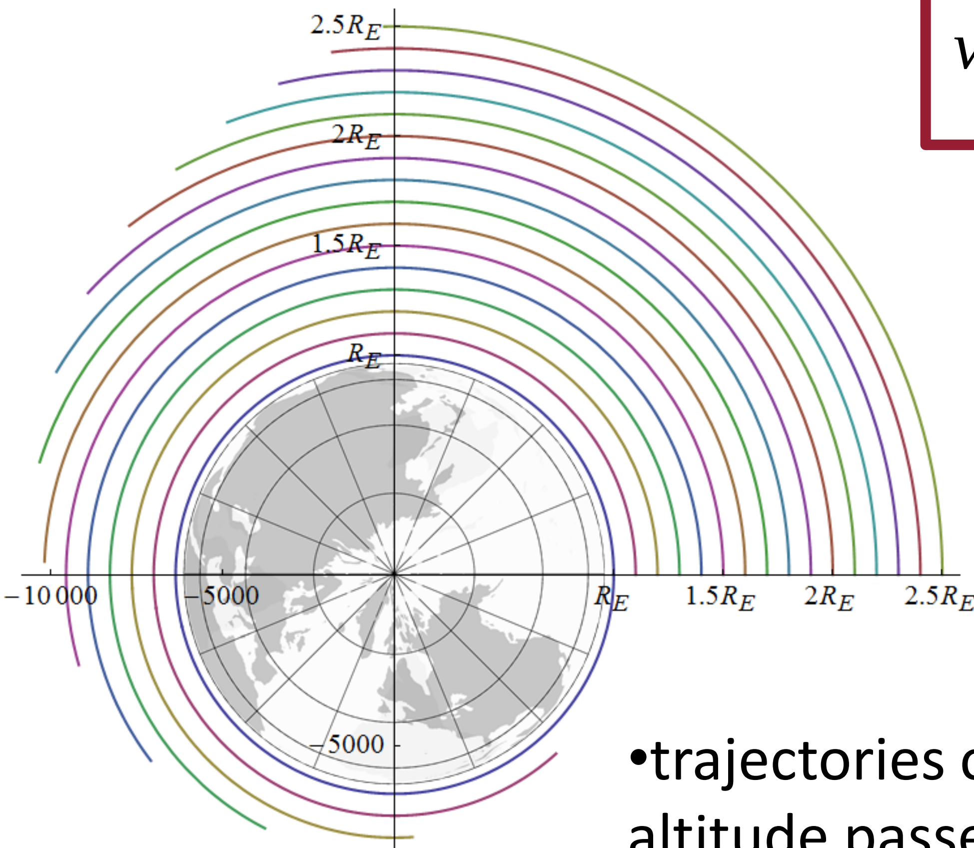
1. The two body problem

Trajectories

- **Circle**: (bounded trajectory) $e = 0$

$$v = \sqrt{\frac{\mu}{r}}$$

$$T = \frac{2\pi}{\sqrt{\mu}} r^{3/2}$$



central body	circ. velocity [km/s]	circ. period [min.]
Earth	7.90	84.48
Moon	1.68	108.36
Mars	3.55	100.19
Sun (surface)	436.7	166.91
Sun (Earths)	29.78	5.26×10^5

•trajectories of satellite in different altitude passed in time T_{Earth}

1. The two body problem

Trajectories

- **Circle**: (bounded trajectory) $e = 0$

$$v = \sqrt{\frac{\mu}{r}}$$

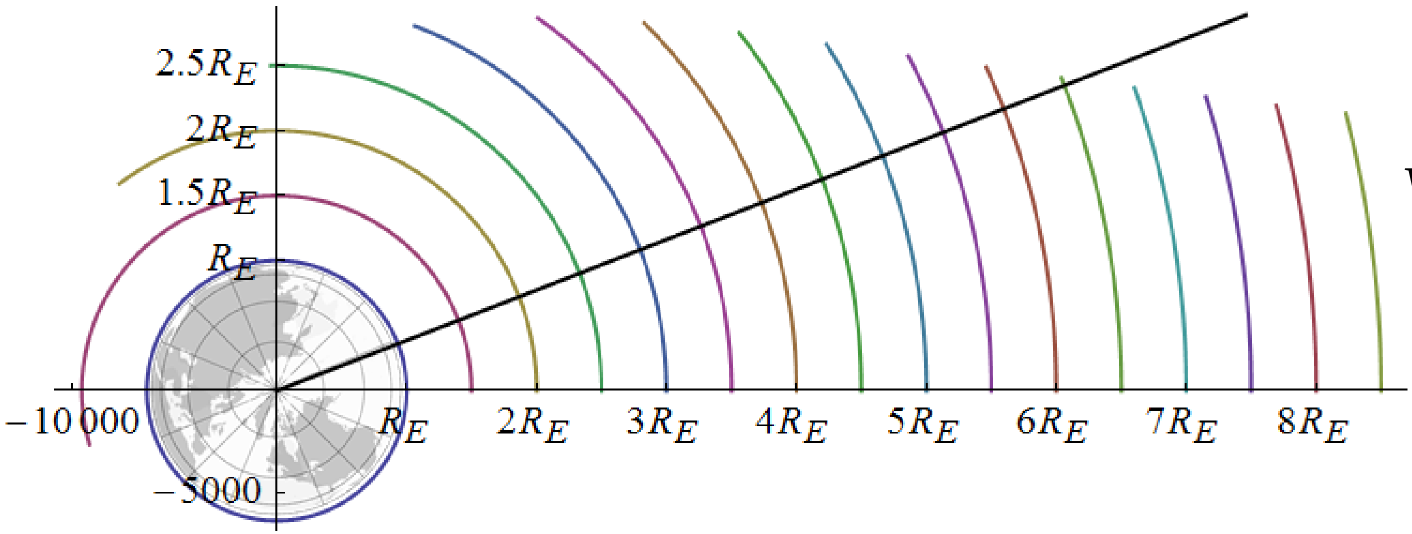
$$T = \frac{2\pi}{\sqrt{\mu}} r^{3/2}$$

•trajectories of satellite in different altitude passed in in time $T_{Earth} = 84.48 \text{ min}$.

$$T_{GEO} = 86164 \text{ s}$$

$$r_{GEO} = 42164 \text{ km}$$

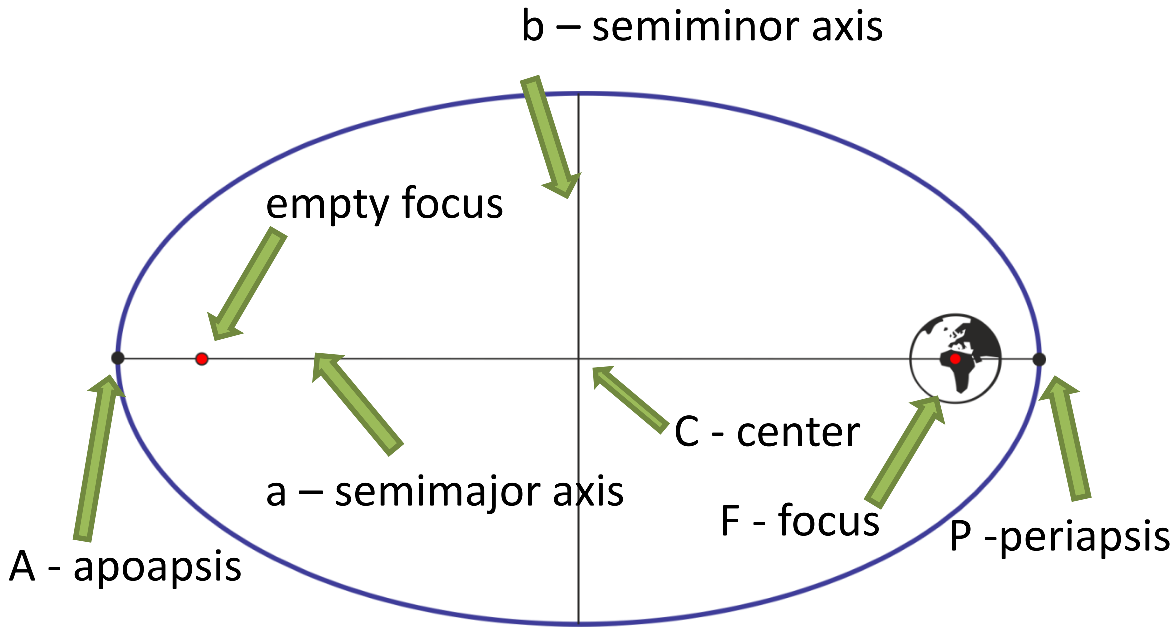
$$v_{GEO} = 3.07 \text{ km/s}$$



1. The two body problem

Trajectories

- **Ellipse**: (bounded trajectory) $0 < e < 1$



1. The two body problem

Trajectories

- **Ellipse**: (bounded trajectory) $0 < e < 1$

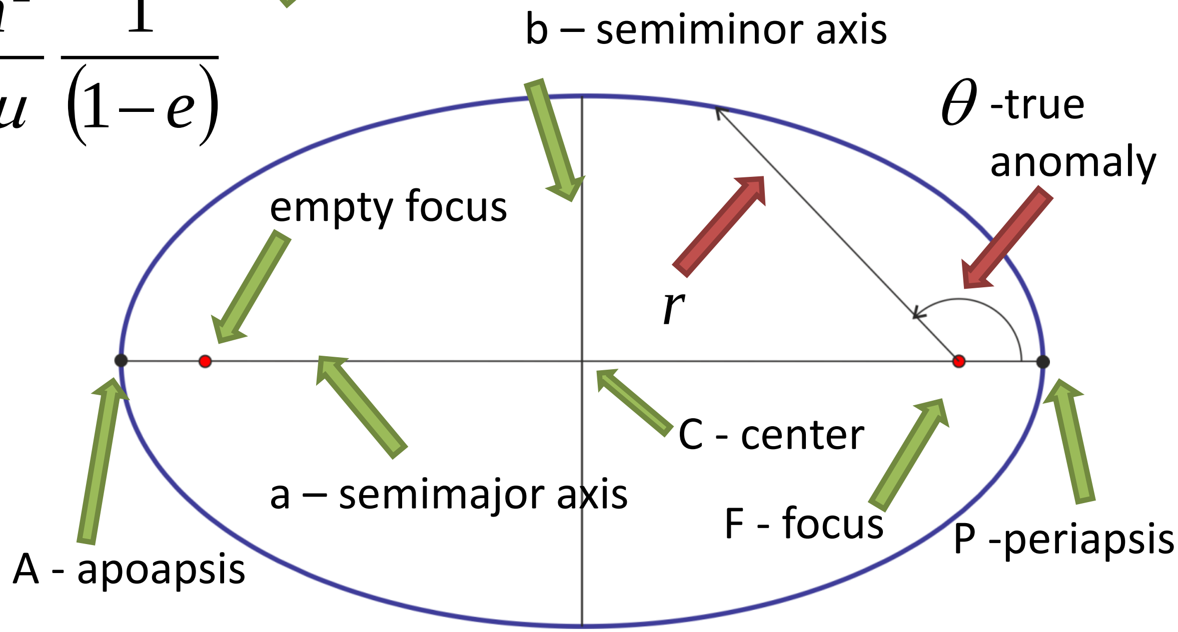
$$r = \frac{h^2}{\mu}\frac{1}{(1+e\cos\theta)}$$

$$r_P = \frac{h^2}{\mu}\frac{1}{(1+e)}$$

$$r_A = \frac{h^2}{\mu}\frac{1}{(1-e)}$$

$$\frac{r_P}{r_A} = \frac{(1-e)}{(1+e)}$$

$$e = \frac{r_A - r_P}{r_A + r_P}$$



1. The two body problem

Trajectories

- **Ellipse**: (bounded trajectory) $0 < e < 1$

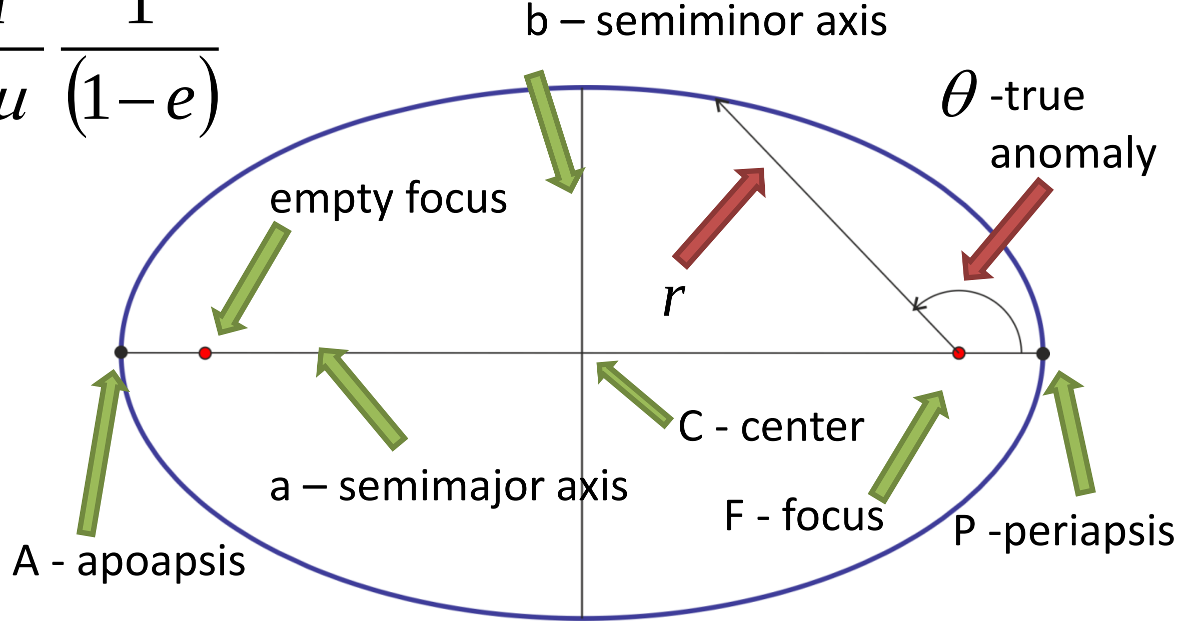
$$r = \frac{h^2}{\mu} \frac{1}{(1 + e\cos\theta)}$$

$$r_P = \frac{h^2}{\mu} \frac{1}{(1+e)}$$

$$r_A = \frac{h^2}{\mu} \frac{1}{(1-e)}$$

$$2a = r_P + r_A$$

$$a = \frac{h^2}{\mu} \frac{1}{1-e^2}$$



1. The two body problem

Trajectories

- **Ellipse**: (bounded trajectory) $0 < e < 1$

$$2a = r_P + r_A$$

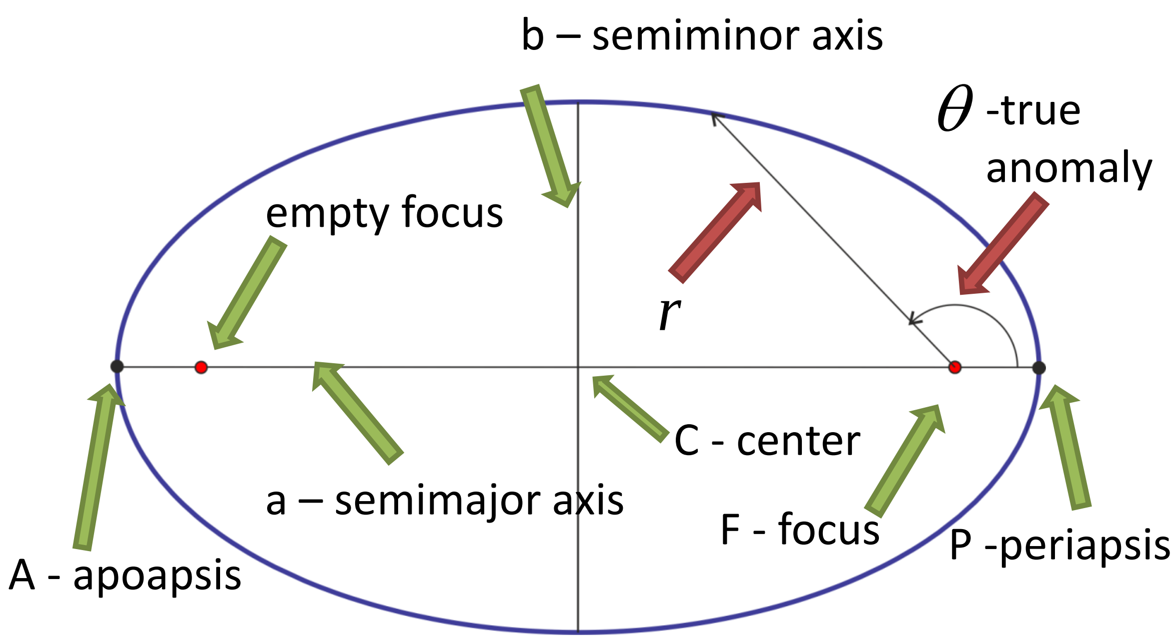
$$e = \frac{r_A - r_P}{r_A + r_P}$$

$$2CF = r_A - r_P$$

$$CF = ea$$

$$b^2 = a^2 - (CF)^2$$

$$b = a\sqrt{(1-e^2)}$$



1. The two body problem

Trajectories

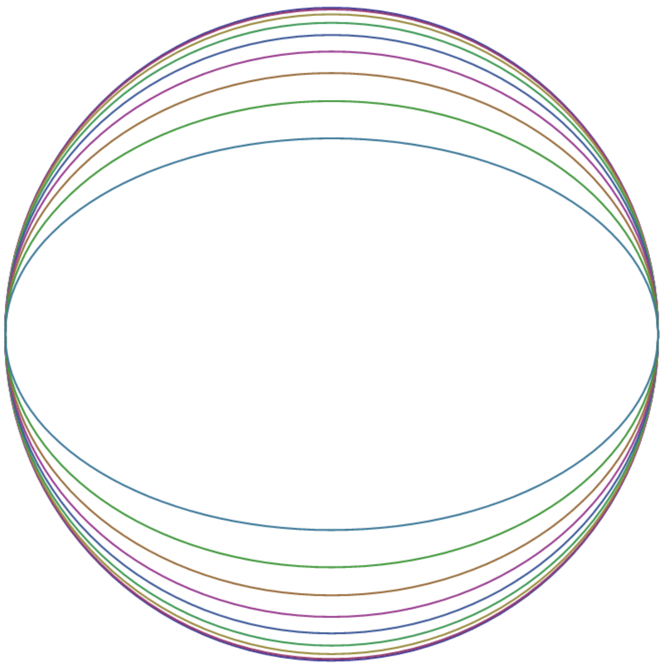
- **Ellipse**: (bounded trajectory) $0 < e < 1$
  - eccentricity
  - flattening

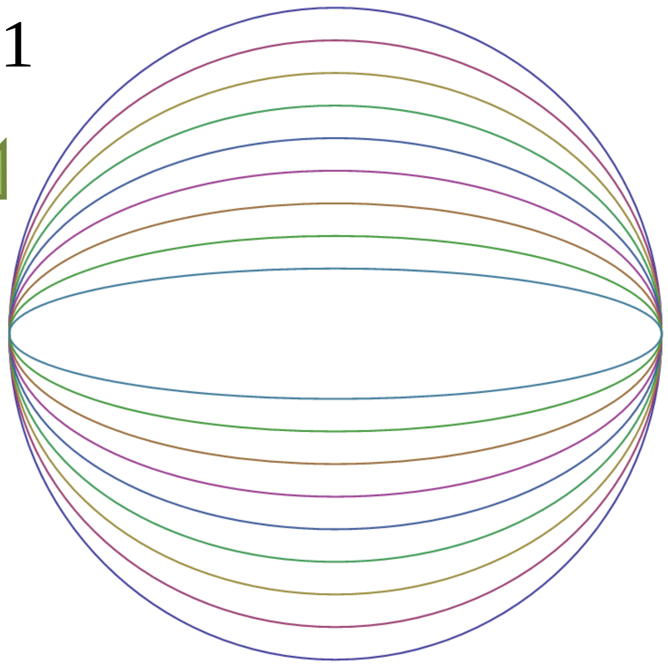
$$e^2 = \frac{a^2 - b^2}{a^2}$$

$$f = 1 - \sqrt{1 - e^2}$$

$$f = \frac{a - b}{a}$$

$\Delta e = 0.1$ $\Delta f = 0.1$





1. The two body problem

Trajectories

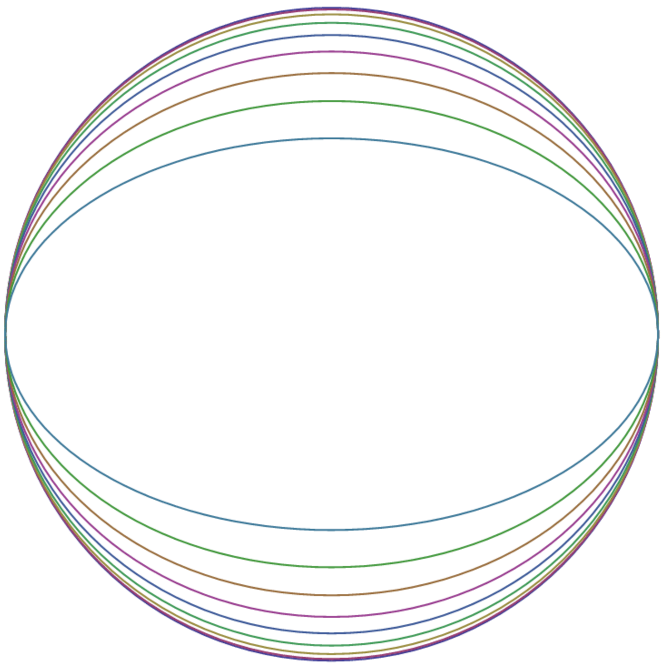
- **Ellipse**: (bounded trajectory) $0 < e < 1$
  - eccentricity
  - flattening

usage: description of orbits

usage: description of planet shape

$\Delta e = 0.1$

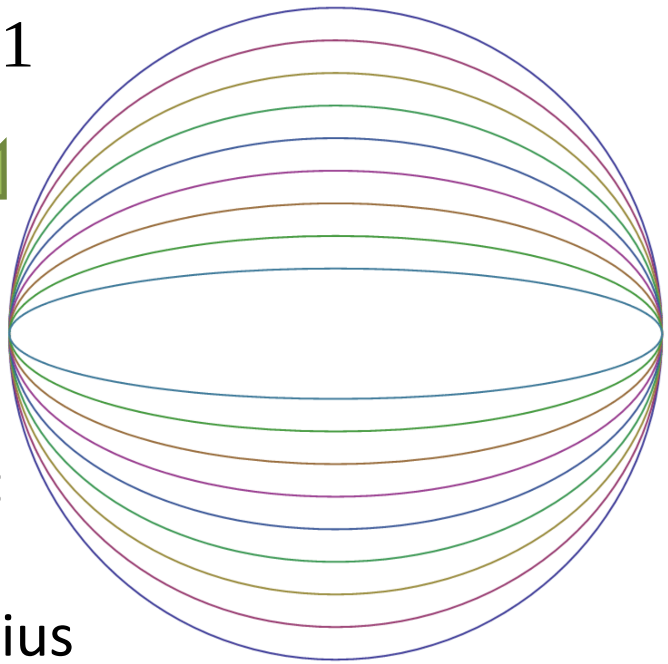
$\Delta f = 0.1$



flattening of Earth:

$1/f = 298.257$

21.4 km diff. in radius



1. The two body problem

Trajectories

- **Ellipse**: (bounded trajectory) $0 < e < 1$
  - equation of orbit $$r = \frac{h^2}{\mu}\frac{1}{(1+e\cos\theta)}$$
  - speed of motion $\varepsilon = \frac{v^2}{2} - \frac{\mu}{r}$ → $$v = \sqrt{-\frac{\mu}{a} + \frac{2\mu}{r}}$$
  - period $T = \frac{2\pi ab}{h}$ → $$T = \frac{2\pi}{\sqrt{\mu}}a^{\frac{3}{2}}$$
  - specific energy $\varepsilon = -\frac{1}{2}\frac{\mu^2}{h^2}(1-e^2)$ → $$\varepsilon = -\frac{1}{2}\frac{\mu}{a}$$

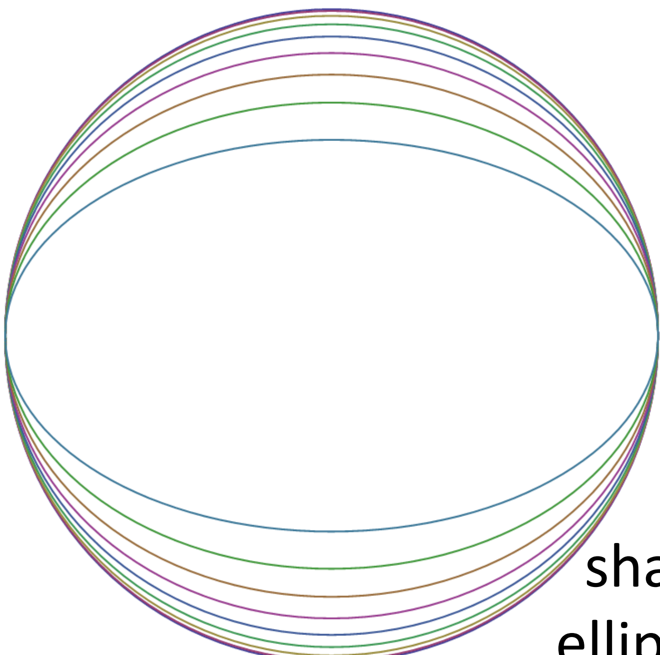
1. The two body problem

- **Ellipse**: (bounded trajectory) $0 < e < 1$
  - ellipses with equal semimajor axis a:

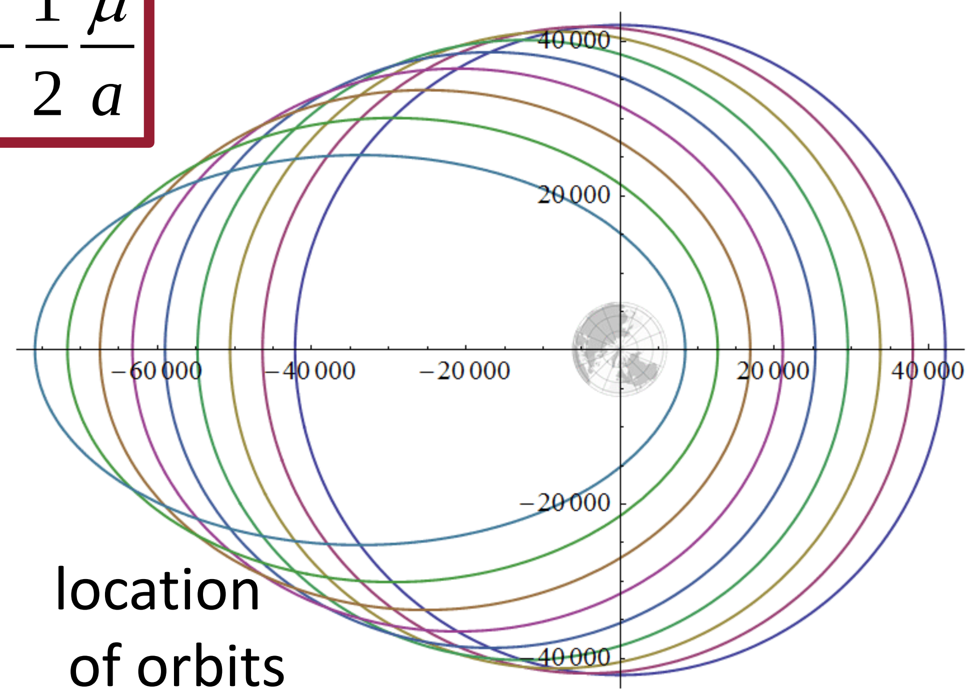
$$T = \frac{2\pi}{\sqrt{\mu}} a^{\frac{3}{2}}$$

$$\varepsilon = -\frac{1}{2}\frac{\mu}{a}$$

equal period and orbital energy



shape of ellipses



location of orbits

1. The two body problem

Trajectories

- **Ellipse**: (bounded trajectory) $0 < e < 1$

  - ellipses with equal semimajor axis a:

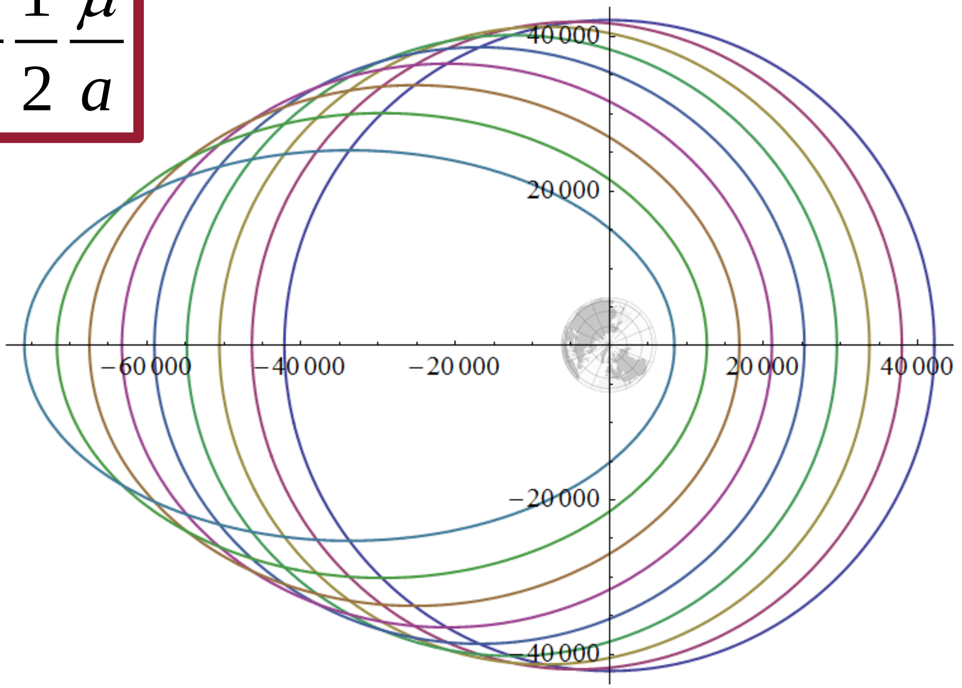
$$T = \frac{2\pi}{\sqrt{\mu}} a^{\frac{3}{2}}$$

$$\varepsilon = -\frac{1}{2}\frac{\mu}{a}$$

equal period and orbital energy

$$v = \sqrt{-\frac{\mu}{a} + \frac{2\mu}{r}}$$

r_p[km]	v_p[km/s]	r_a[km]	v_a[km/s]
42164	3.07	42164	3.07
29514.8	4.19	54813.2	2.25
16865.6	6.15	67462.4	1.53
8432.8	9.22	75895.2	1.02



1. The two body problem

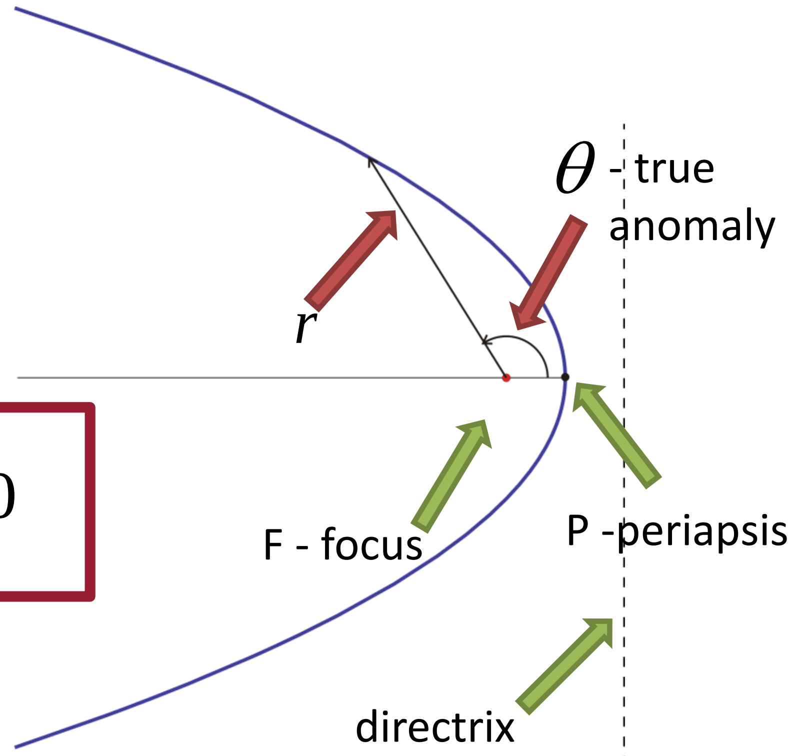
Trajectories

- **Parabola**: (open trajectory) $e = 1$

- equation of orbit → $$r = \frac{h^2}{\mu}\frac{1}{(1+1\cos\theta)}$$

- speed of motion $$\varepsilon = \frac{v^2}{2} - \frac{\mu}{r}$$ → $$v = \sqrt{\frac{2\mu}{r}}$$

- specific energy $$\varepsilon = -\frac{1}{2}\frac{\mu^2}{h^2}(1-e^2)$$ → $$\varepsilon = 0$$

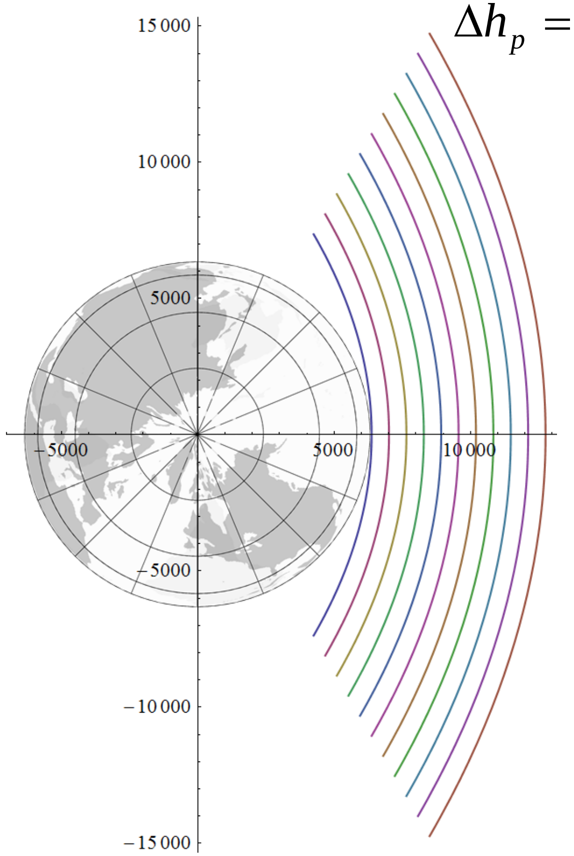


1. The two body problem

Trajectories

- **Parabola**: (open trajectory) $e = 1$

$$\Delta h_p = \frac{1}{10} R_{Earth}$$



$$v = \sqrt{\frac{2\mu}{r}}$$

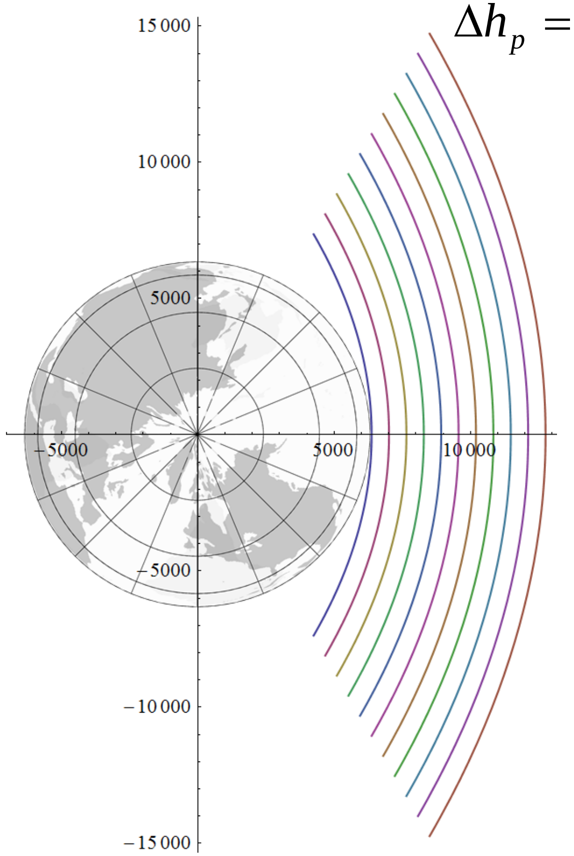
central body	esc. velocity [km/s]
Earth	11.18
Moon	2.37
Mars	5.02
Sun (surface)	617.5
Sun (Earths)	42.12

1. The two body problem

Trajectories

- **Parabola**: (open trajectory) $e = 1$

$$\Delta h_p = \frac{1}{10} R_{Earth}$$



•trajectories of satellite in different $\Delta h_p = \frac{1}{10} R_{Earth}$

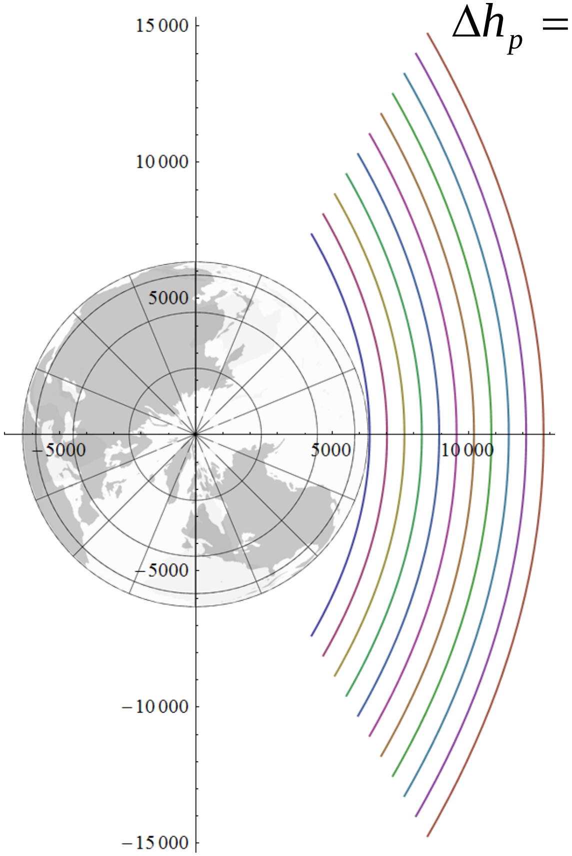
Δh_p [km] (Earth)	v_p[km/s]
0	11.18
637.8	10.66
1275.6	10.20
1913.4	9.80
2551.2	9.44
3189.0	9.12
3826.8	8.83

1. The two body problem

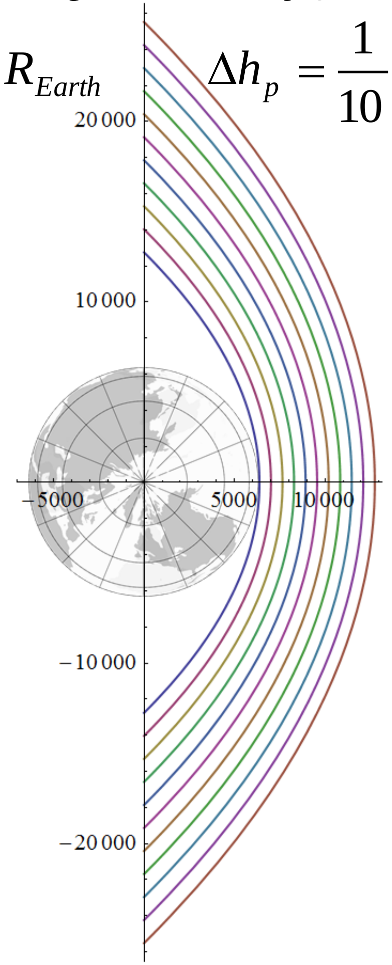
Trajectories

- **Parabola**: (open trajectory) $e=1$

$$\Delta h_p = \frac{1}{10} R_{Earth}$$



$$\Delta h_p = \frac{1}{10} R_{Earth}$$

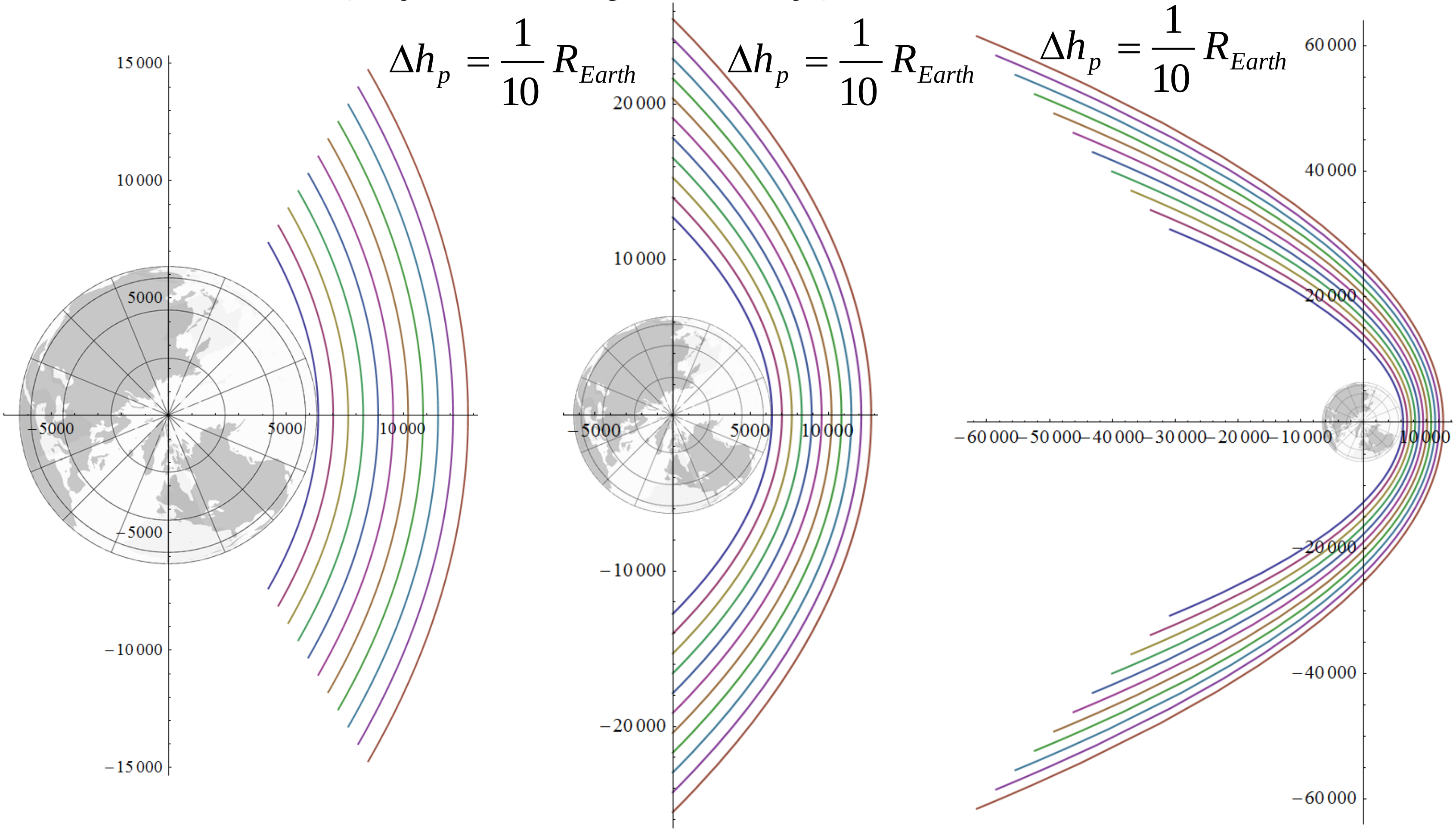


•trajectories of satellite in different $\Delta h_p = \frac{1}{10} R_{Earth}$

1. The two body problem

Trajectories

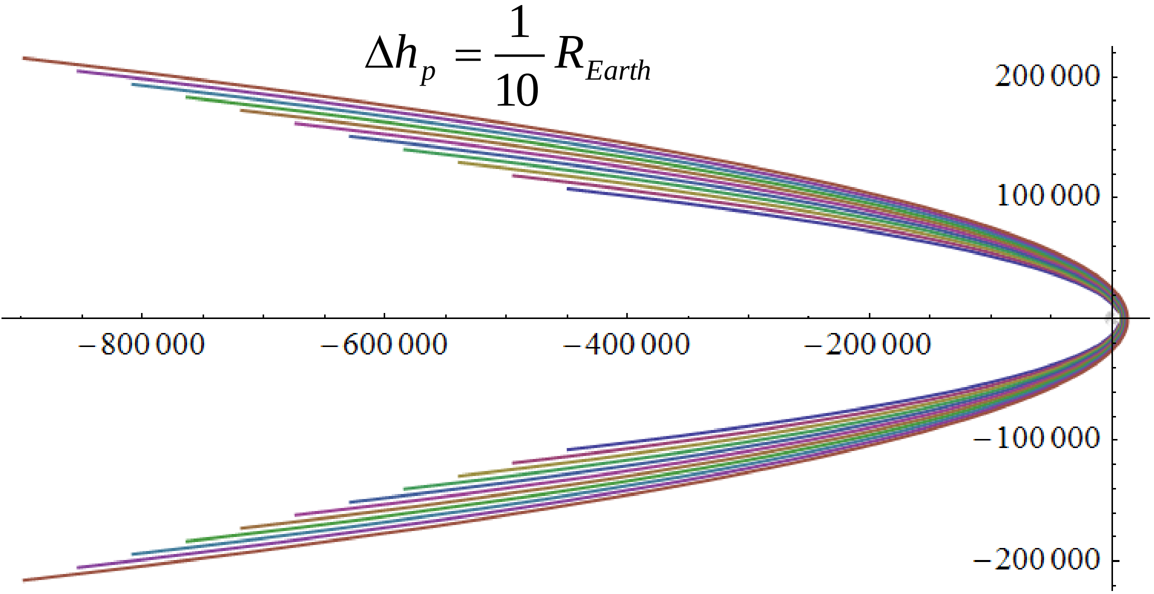
- **Parabola**: (open trajectory) $e = 1$

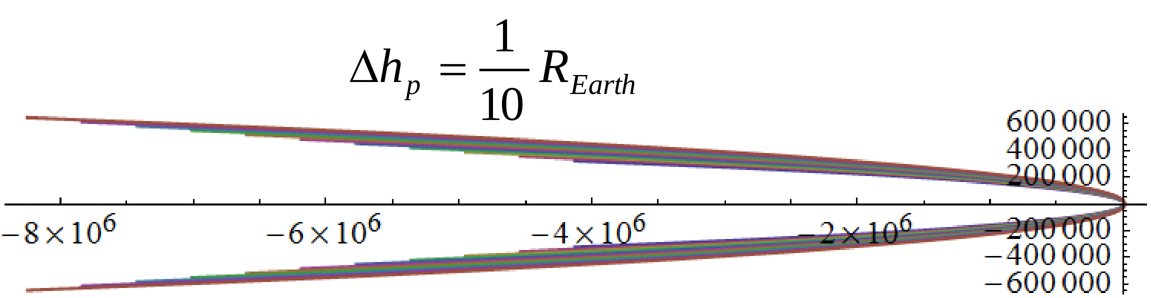


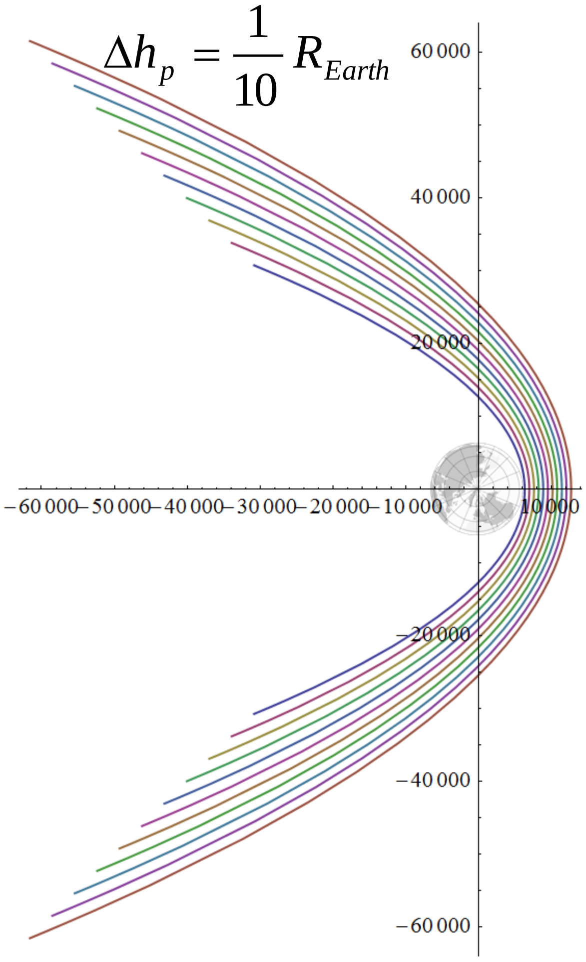
1. The two body problem

Trajectories

- **Parabola**: (open trajectory) $e = 1$



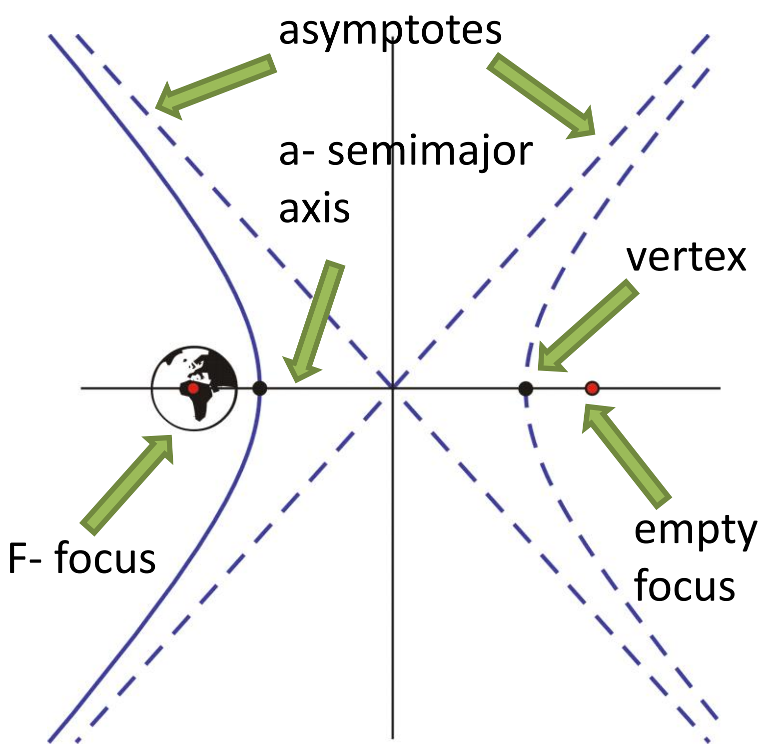




1. The two body problem

Trajectories

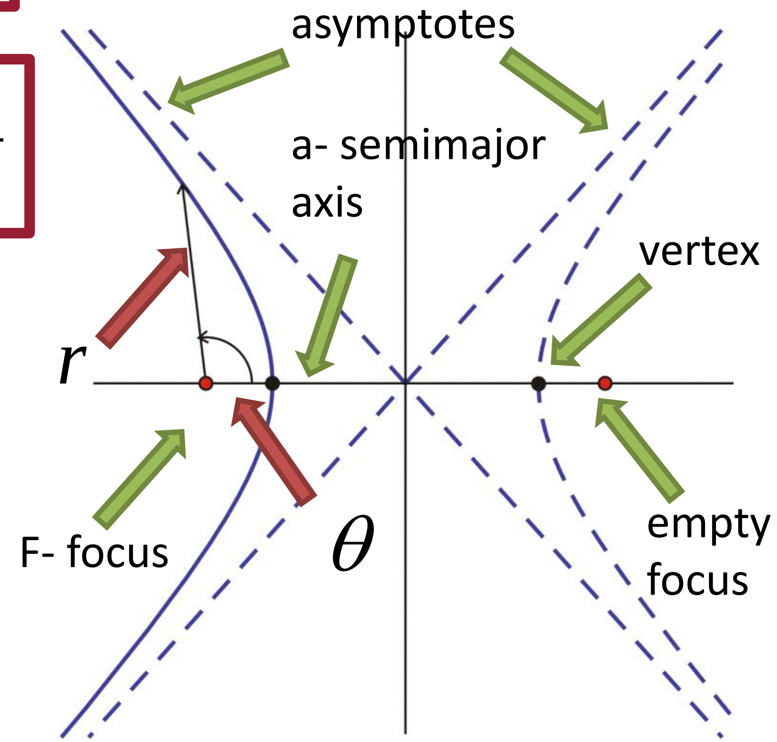
- **Hyperbola**: (open trajectory) $e > 1$



1. The two body problem

Trajectories

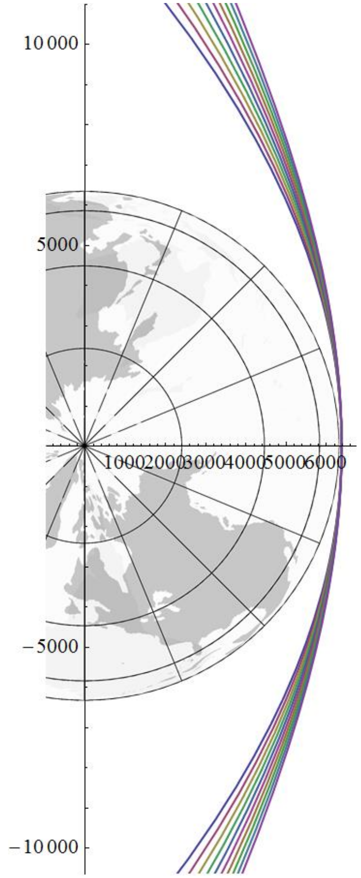
- **Hyperbola**: (open trajectory) $e > 1$
  - equation of orbit $$r = \frac{h^2}{\mu} \frac{1}{(1 + e\cos\theta)}$$
  - speed of motion $\varepsilon = \frac{v^2}{2} - \frac{\mu}{r}$ $$v^2 = \frac{\mu}{a} + \frac{2\mu}{r}$$
  - specific energy $$\varepsilon = \frac{\mu}{2a}$$ $$\varepsilon = -\frac{1}{2} \frac{\mu^2}{h^2} (1 - e^2)$$



1. The two body problem

Trajectories

- **Hyperbola**: (open trajectory) $e > 1$



•trajectories of satellite with different $\Delta e = 0.1$

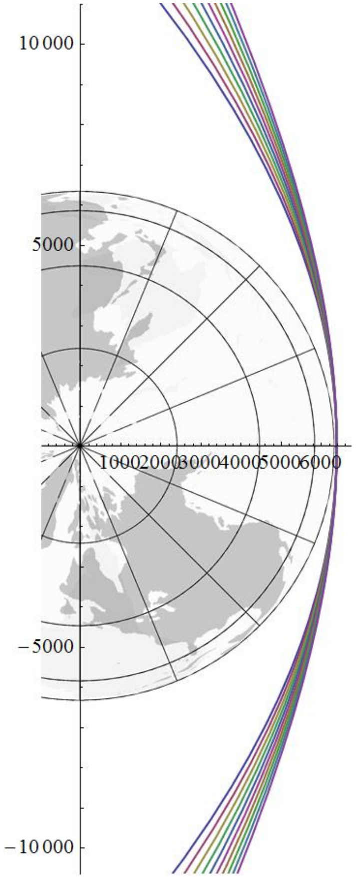
• periapsis: $r_p = R_{Earth}$

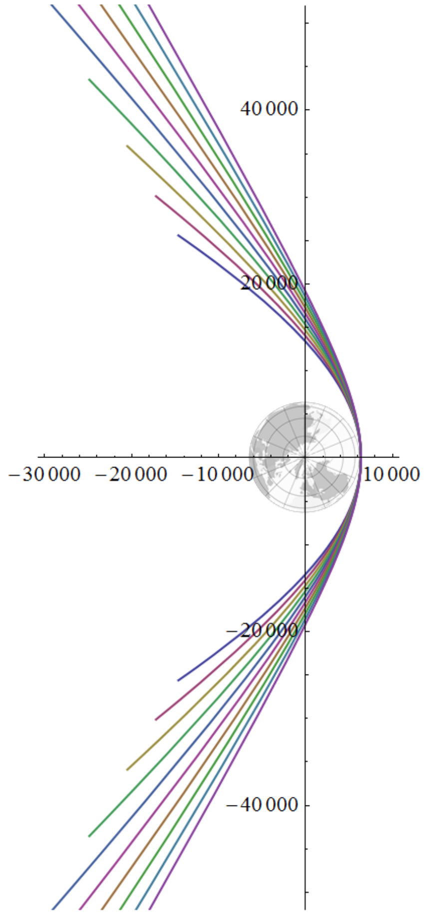
e	v_p[km/s]
1.1	11.45
1.2	11.72
1.3	11.98
1.4	12.24
1.5	12.49
1.6	12.74
1.7	12.99

1. The two body problem

Trajectories

- **Hyperbola**: (open trajectory) $e > 1$





- trajectories of satellite with different $\Delta e = 0.1$
- periapsis: $r_p = R_{Earth}$

e	v_p[km/s]
1.1	11.45
1.2	11.72
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1.6	12.74
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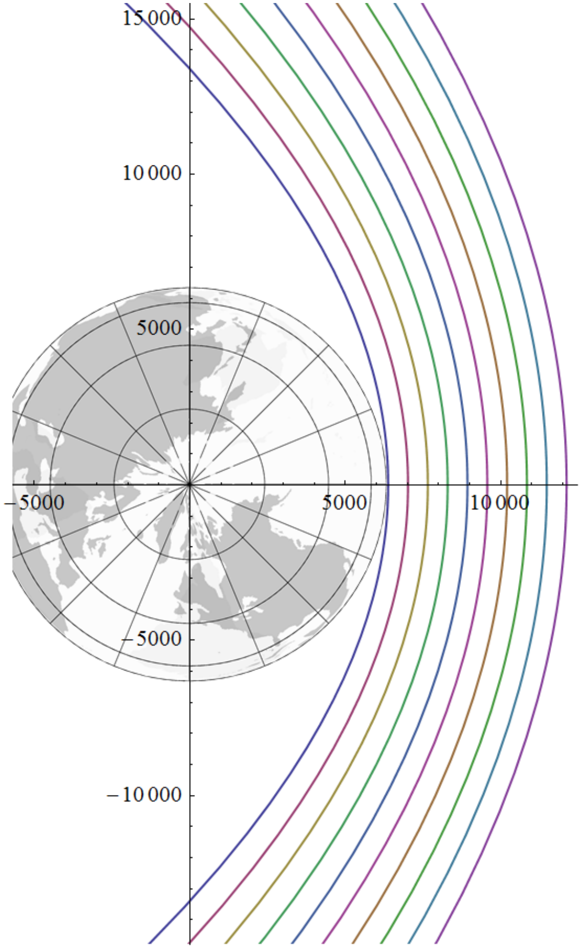
1. The two body problem

Trajectories

- **Hyperbola**: (open trajectory) $e > 1$



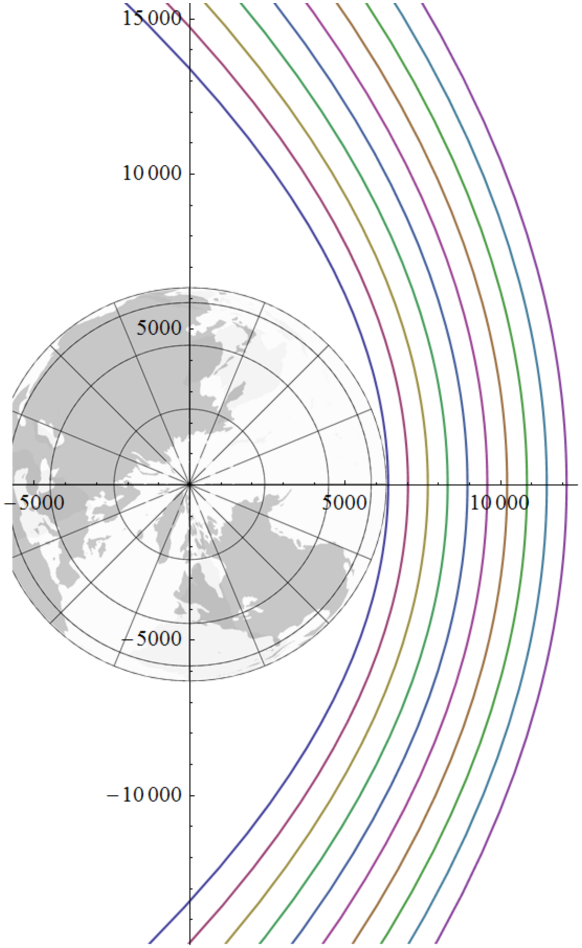
- trajectories of satellite with different alt. $\Delta h_p = 0.1 R_{Earth}$
- eccentricity: $e = 1.1$

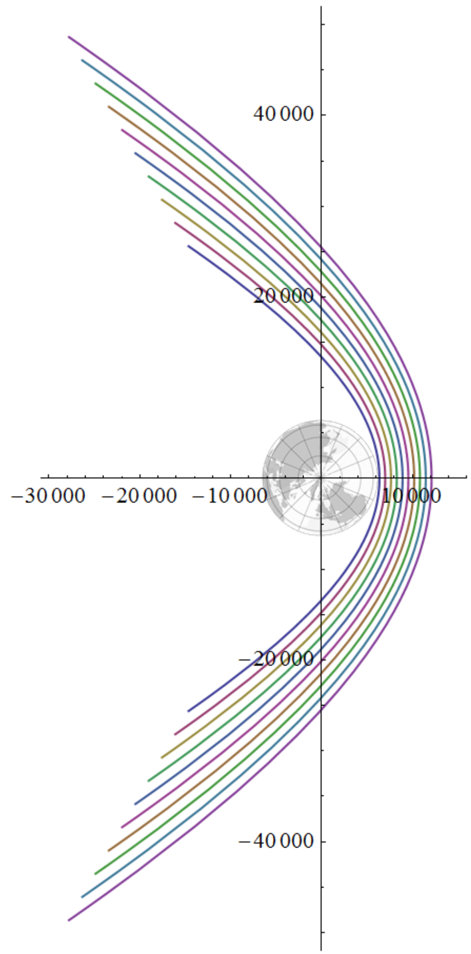
h_p	v_p[km/s]
$0.1 \times R_{Earth}$	11.45
$0.2 \times R_{Earth}$	10.92
$0.3 \times R_{Earth}$	10.45
$0.4 \times R_{Earth}$	10.04
$0.5 \times R_{Earth}$	9.68
$0.6 \times R_{Earth}$	9.35
$0.7 \times R_{Earth}$	9.05

1. The two body problem

Trajectories

- **Hyperbola**: (open trajectory) $e > 1$





•trajectories of satellite with different alt. $\Delta h_p = 0.1 R_{Earth}$

• eccentricity: $e = 1.1$

h_p	v_p[km/s]
$0.1 \text{x} R_{Earth}$	11.45
$0.2 \text{x} R_{Earth}$	10.92
$0.3 \text{x} R_{Earth}$	10.45
$0.4 \text{x} R_{Earth}$	10.04
$0.5 \text{x} R_{Earth}$	9.68
$0.6 \text{x} R_{Earth}$	9.35
$0.7 \text{x} R_{Earth}$	9.05

1. The two body problem

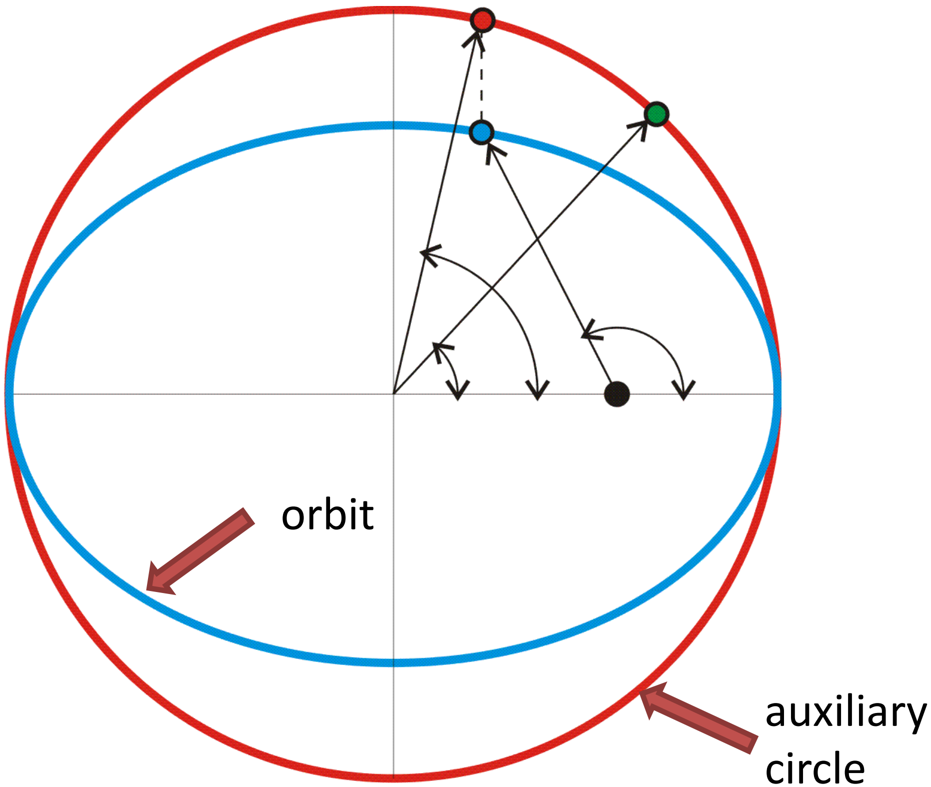
Time and position

- Two cases can be investigated:
  - time as a function of position
  - position as a function of time
- Only ellipse orbit is presented, but similar expressions can be derived for all trajectories

1. The two body problem

Time and position

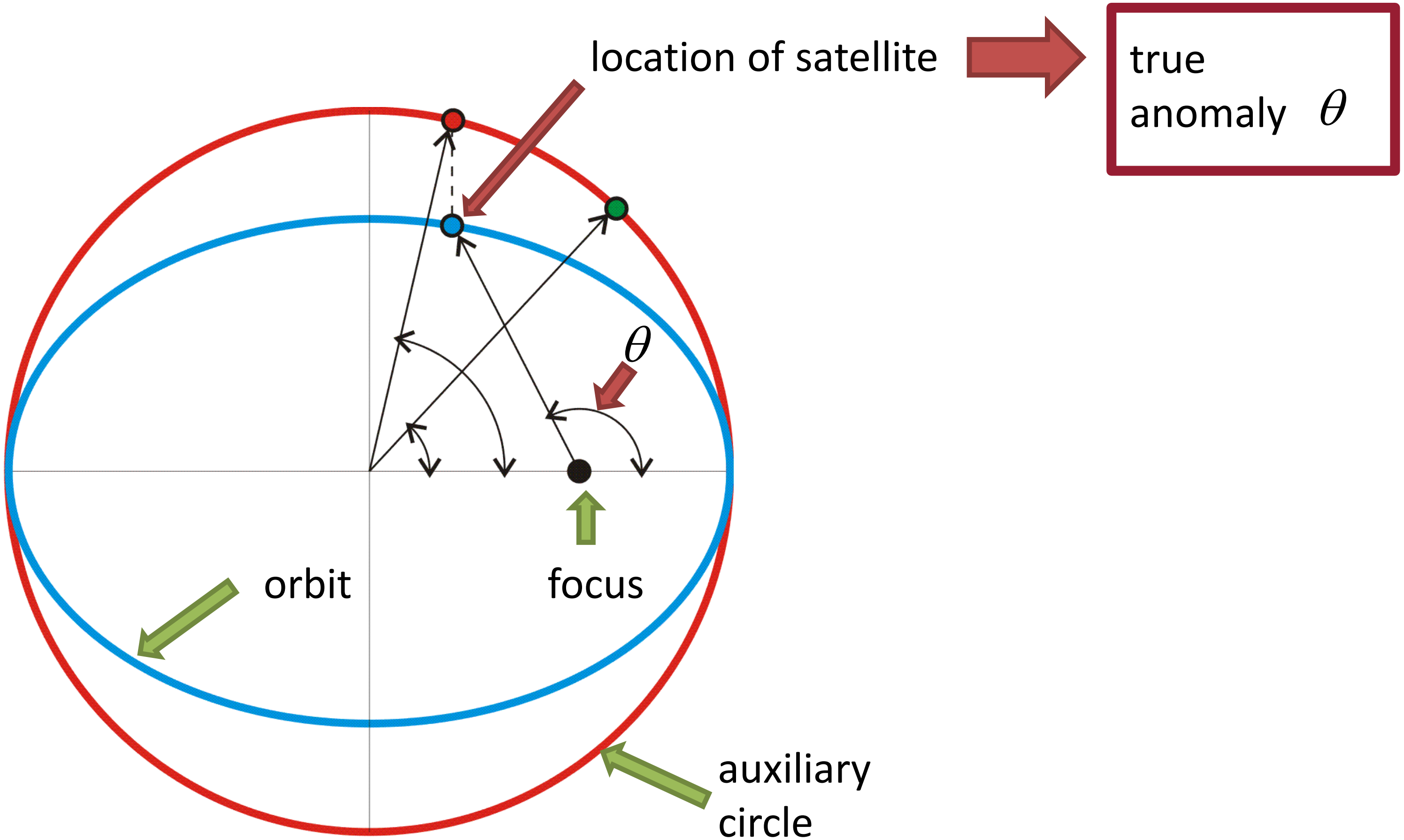
- True, Mean and Eccentric anomalies



1. The two body problem

Time and position

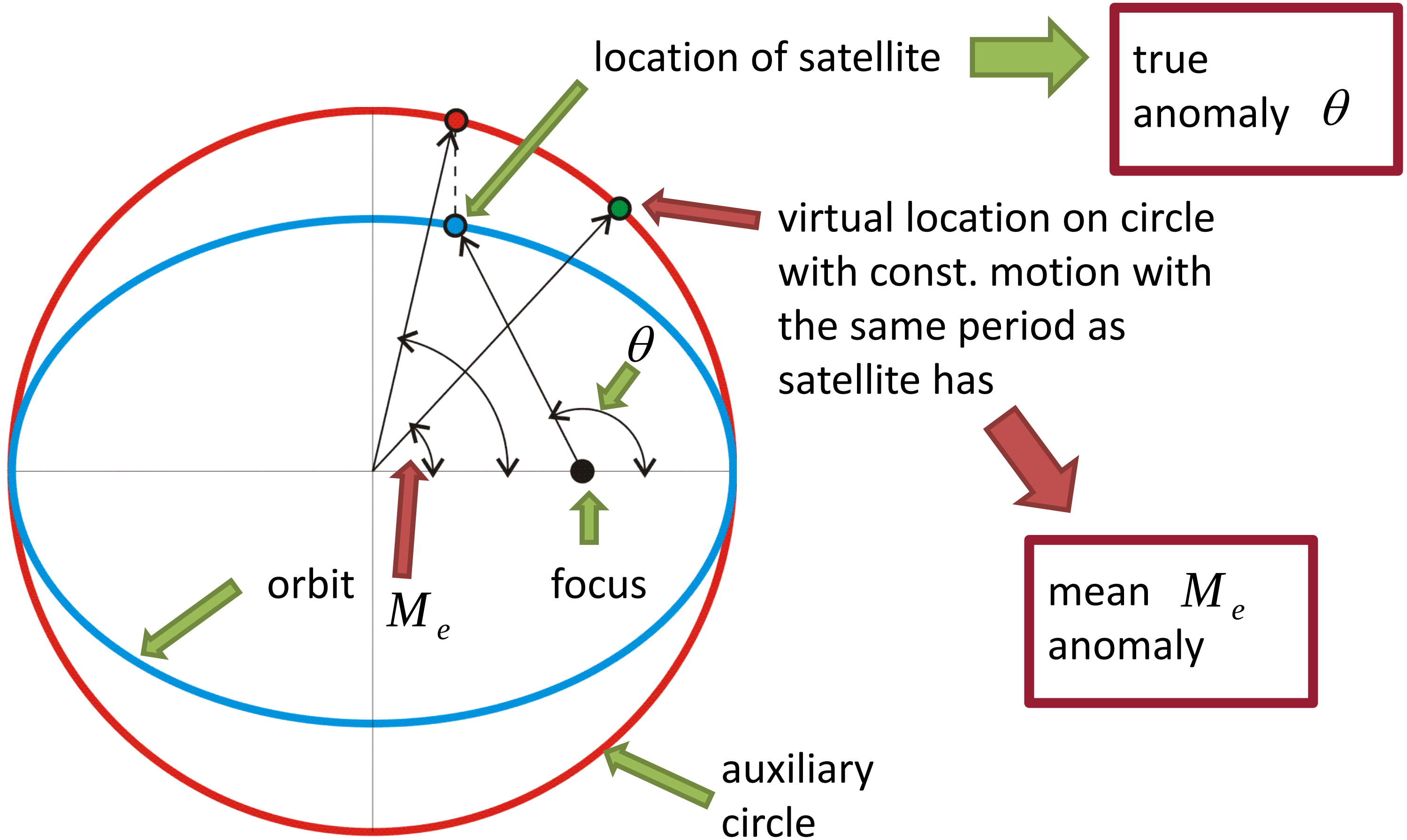
- True, Mean and Eccentric anomalies



1. The two body problem

Time and position

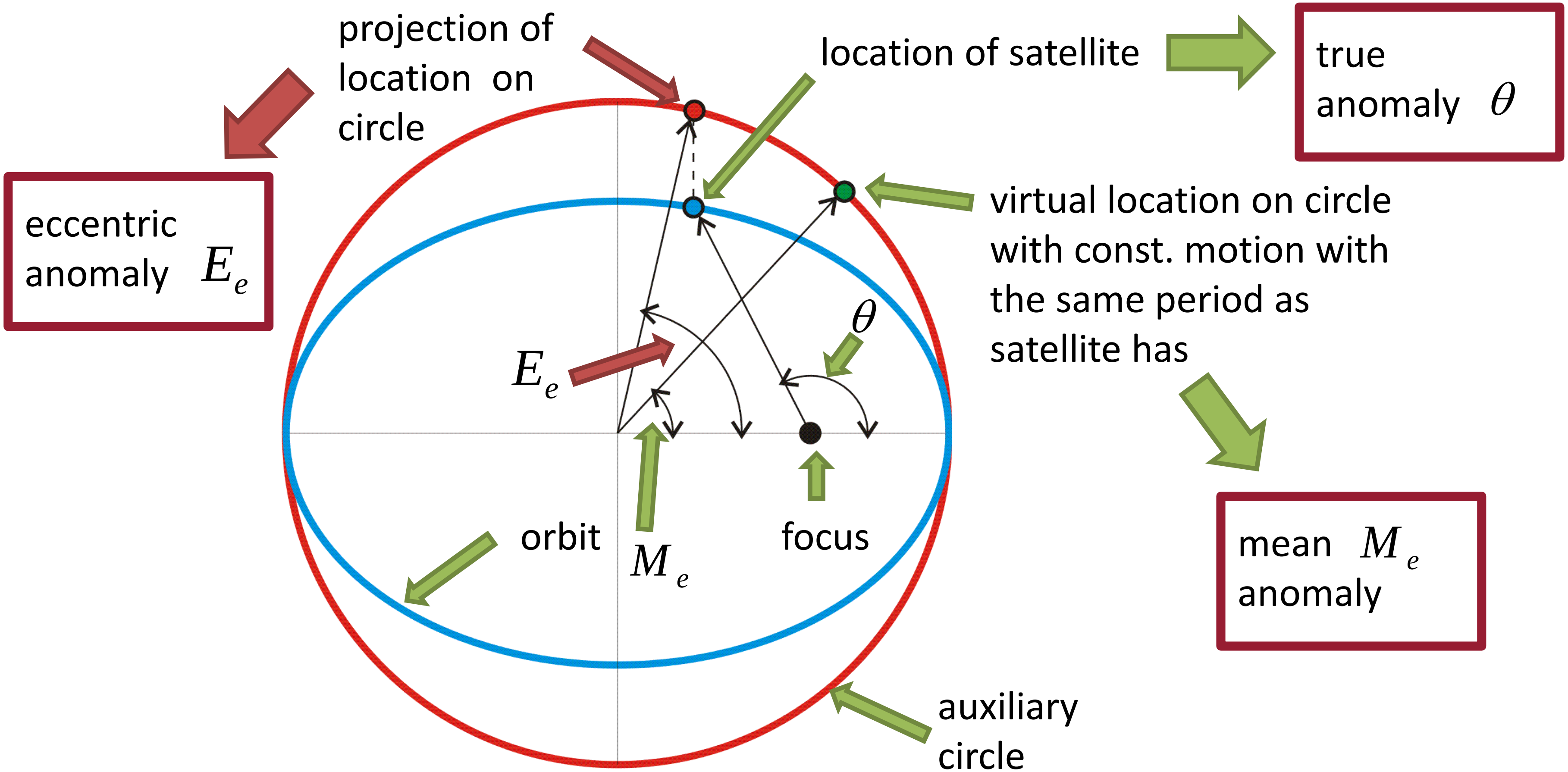
- True, Mean and Eccentric anomalies



1. The two body problem

Time and position

- True, Mean and Eccentric anomalies



1. The two body problem

Time and position

- Time as a function of position

• using mean anomaly

$$h = \dot{\theta} r^2 = \frac{d\theta}{dt} r^2$$

1. The two body problem

Time and position

- Time as a function of position

- using mean anomaly

$$h = \dot{\theta} r^2 = \frac{d\theta}{dt} r^2$$

$$dt = \frac{1}{h} r^2 d\theta$$

1. The two body problem

Time and position

- Time as a function of position
  - using mean anomaly

$$h = \dot{\theta} r^2 = \frac{d\theta}{dt} r^2$$

$$dt = \frac{1}{h} r^2 d\theta$$

$$r = \frac{h^2}{\mu} \frac{1}{(1 + e\cos\theta)}$$

1. The two body problem

Time and position

- Time as a function of position
  - using mean anomaly

$$h = \dot{\theta} r^2 = \frac{d\theta}{dt} r^2$$

$$dt = \frac{1}{h} r^2 d\theta \quad \Rightarrow \quad dt = \frac{h^3}{\mu^2} \frac{d\theta}{(1 + e\cos\theta)^2}$$

$$r = \frac{h^2}{\mu} \frac{1}{(1 + e\cos\theta)}$$

1. The two body problem

Time and position

- Time as a function of position
  - using mean anomaly

$$h = \dot{\theta} r^2 = \frac{d\theta}{dt} r^2$$

$$dt = \frac{1}{h} r^2 d\theta \quad \Rightarrow \quad dt = \frac{h^3}{\mu^2} \frac{d\theta}{(1 + e\cos\theta)^2}$$

$$r = \frac{h^2}{\mu} \frac{1}{(1 + e\cos\theta)}$$

$$t - t_p = \frac{h^3}{\mu^2} \int_0^{\theta} \frac{d\vartheta}{(1 + e\cos\vartheta)^2}$$

1. The two body problem

Time and position

- Time as a function of position - **ellipse** $0 < e < 1$

• using mean anomaly

$$t - t_p = \frac{h^3}{\mu^2} \int_0^{\theta} \frac{d\vartheta}{(1 + e\cos\vartheta)^2}$$

$t_p = 0$

1. The two body problem

Time and position

- Time as a function of position - **ellipse** $0 < e < 1$

- using mean anomaly

$$t - t_p = \frac{h^3}{\mu^2} \int_0^{\theta} \frac{d\vartheta}{(1 + e\cos\vartheta)^2}$$

$$t_p = 0$$

$$\int_0^{\theta} \frac{d\vartheta}{(1 + e\cos\vartheta)^2} = \frac{1}{(1 - e^2)^{3/2}} \left[2\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}} \tan\frac{\theta}{2} \right) - e\frac{\sqrt{1-e^2}\sin\theta}{1 + e\cos\theta} \right]$$

1. The two body problem

Time and position

- Time as a function of position - **ellipse** $0 < e < 1$

- using mean anomaly

$$t - t_p = \frac{h^3}{\mu^2} \int_0^{\theta} \frac{d\vartheta}{(1 + e\cos\vartheta)^2}$$

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$$\int_0^{\theta} \frac{d\vartheta}{(1 + e\cos\vartheta)^2} = \frac{1}{(1 - e^2)^{3/2}} M_e$$

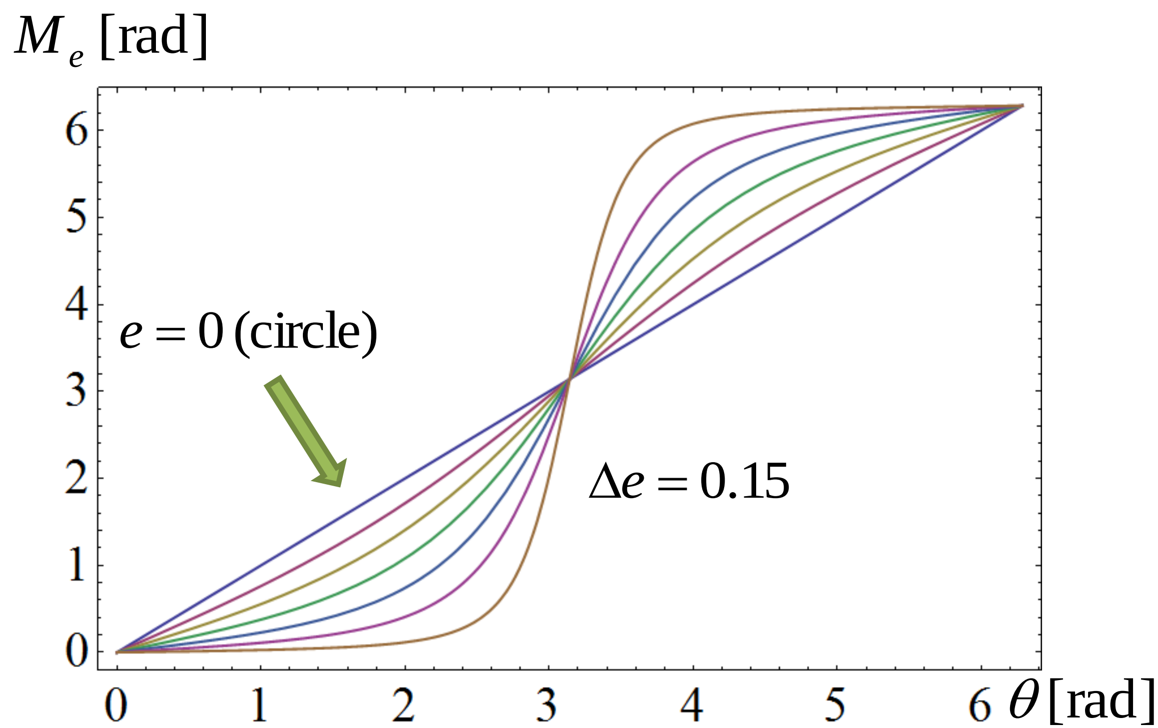
1. The two body problem

Time and position

- Time as a function of position - **ellipse** $0 < e < 1$

• using mean anomaly

$$t = \frac{h^3}{\mu^2} \frac{1}{\left(1-e^2\right)^{3/2}} M_e$$



$$M_e = \left[2\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}} \tan\frac{\theta}{2} \right) - e\frac{\sqrt{1-e^2}\sin\theta}{1+e\cos\theta} \right]$$

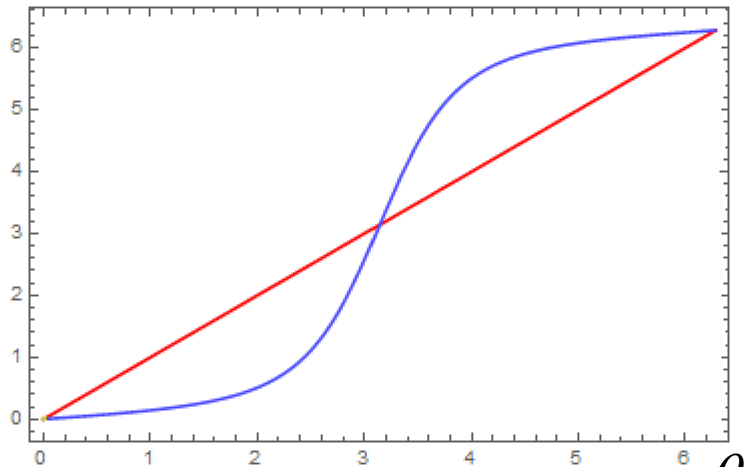
1. The two body problem

Time and position

- Time as a function of position - **ellipse** $0 < e < 1$

• using mean anomaly

M_e [rad]



θ [rad]

$e = 0.576$

$$M_e = \left[2\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2}\right) - e\frac{\sqrt{1-e^2}\sin\theta}{1+e\cos\theta} \right]$$

1. The two body problem

Time and position

- Time as a function of position - **ellipse** $0 < e < 1$

• using eccentric anomaly

$$t = \frac{h^3}{\mu^2} \frac{1}{\left(1-e^2\right)^{3/2}} M_e$$

E_e $\qquad$ $\sin E_e$

$$M_e = \left[2\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}} \tan\frac{\theta}{2} \right) - e \frac{\sqrt{1-e^2}\sin\theta}{1+e\cos\theta} \right]$$

1. The two body problem

Time and position

- Time as a function of position - **ellipse** $0 < e < 1$

- using eccentric anomaly

$$t = \frac{h^3}{\mu^2} \frac{1}{\left(1-e^2\right)^{3/2}} M_e$$

- Kepler's equation

$$M_e = \left(E - e\sin E\right)$$

E_e $\qquad$ $\sin E_e$

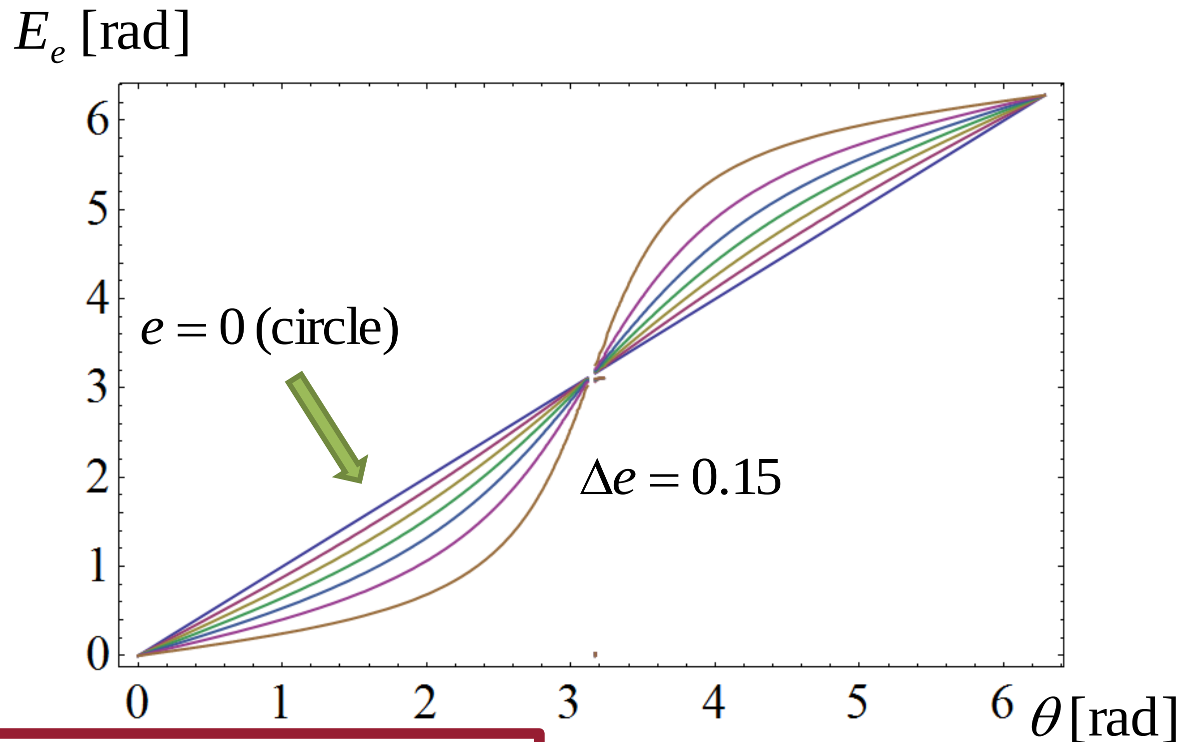
$$M_e = \left[2\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}} \tan\frac{\theta}{2} \right) - e \frac{\sqrt{1-e^2}\sin\theta}{1+e\cos\theta} \right]$$

1. The two body problem

Time and position

- Time as a function of position - **ellipse** $0 < e < 1$

• using eccentric anomaly



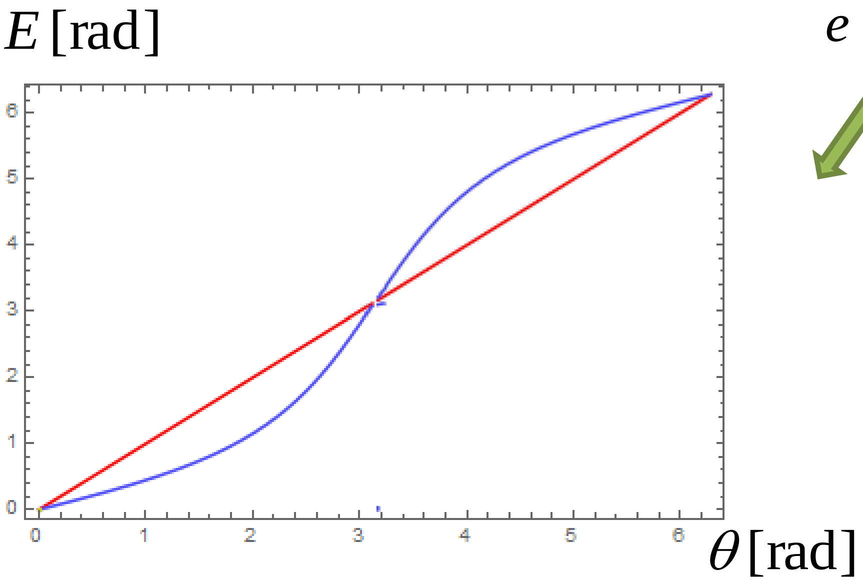
$$E_e = 2\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2}\right)$$

1. The two body problem

Time and position

- Time as a function of position - **ellipse** $0 < e < 1$

• using eccentric anomaly



$e = 0.576$

$$E_e = 2\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2}\right)$$

1. The two body problem

Time and position

- Time as a function of position - **ellipse** $0 < e < 1$

- defined θ

$$E_e = 2\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2}\right)$$

1. The two body problem

Time and position

- Time as a function of position - **ellipse** $0 < e < 1$

- defined θ

$$E_e = 2\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2}\right)$$

$$M_e = \left(E - e\sin E\right)$$

1. The two body problem

Time and position

- Time as a function of position - **ellipse** $0 < e < 1$

- defined θ

$$E_e = 2\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2}\right)$$

$$M_e = (E - e\sin E)$$

$$t = \frac{h^3}{\mu^2}\frac{1}{\left(1-e^2\right)^{3/2}}M_e$$

1. The two body problem

Time and position

- Time as a function of position - **ellipse** $0 < e < 1$

- defined θ

$$E_e = 2\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2}\right)$$

$$M_e = \left(E - e\sin E\right)$$

$$t = \frac{h^3}{\mu^2}\frac{1}{\left(1-e^2\right)^{3/2}}M_e \Rightarrow t = \frac{T}{2\pi}M_e$$

1. The two body problem

Time and position

- Time as a function of position - **ellipse** $0 < e < 1$

- defined θ

$$E_e = 2\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2}\right)$$

$$M_e = \left(E - e\sin E\right)$$

$$t = \frac{h^3}{\mu^2}\frac{1}{\left(1-e^2\right)^{3/2}}M_e$$

$$t = \frac{T}{2\pi}M_e$$

$$n = \frac{2\pi}{T}$$

- average angular velocity

$$M_e = nt$$

1. The two body problem

Time and position

- Position as a function of time - **ellipse** $0 < e < 1$

- defined t

↓

$$M_e = \frac{2\pi}{T} t$$

1. The two body problem

Time and position

- Position as a function of time - **ellipse** $0 < e < 1$

- defined t

$$M_e = \frac{2\pi}{T} t$$

$$M_e = \left(E_e - e \sin E_e\right)$$

1. The two body problem

Time and position

- Position as a function of time - **ellipse** $0 < e < 1$

- defined t

$$M_e = \frac{2\pi}{T} t$$

$$M_e = \left(E_e - e \sin E_e \right)$$

- E_e must be computed numerically

1. The two body problem

Time and position

- Position as a function of time - **ellipse** $0 < e < 1$

• defined t

$$M_e = \frac{2\pi}{T} t$$

$$M_e = \left(E_e - e \sin E_e\right)$$

• E_e must be computed numerically

• the problem of finding true anomaly for defined time is called Kepler's problem

$$\theta = 2 \tan^{-1}\left(\sqrt{\frac{1+e}{1-e}} \tan\frac{E_e}{2}\right)$$

2. Orbits in three dimensions

- Frame of reference
- Earth-based systems
- Orbital elements
- Calculation of elements

2. Orbits in three dimensions

Frame of reference

- to describe orbits in three dimensions, the coordinate system in frame of reference must be defined
- Newton laws are valid in inertial frame of reference
- practically only pseudoinertial frame of reference can be considered
- coordinate system is formed in considered frame of reference

2. Orbits in three dimensions

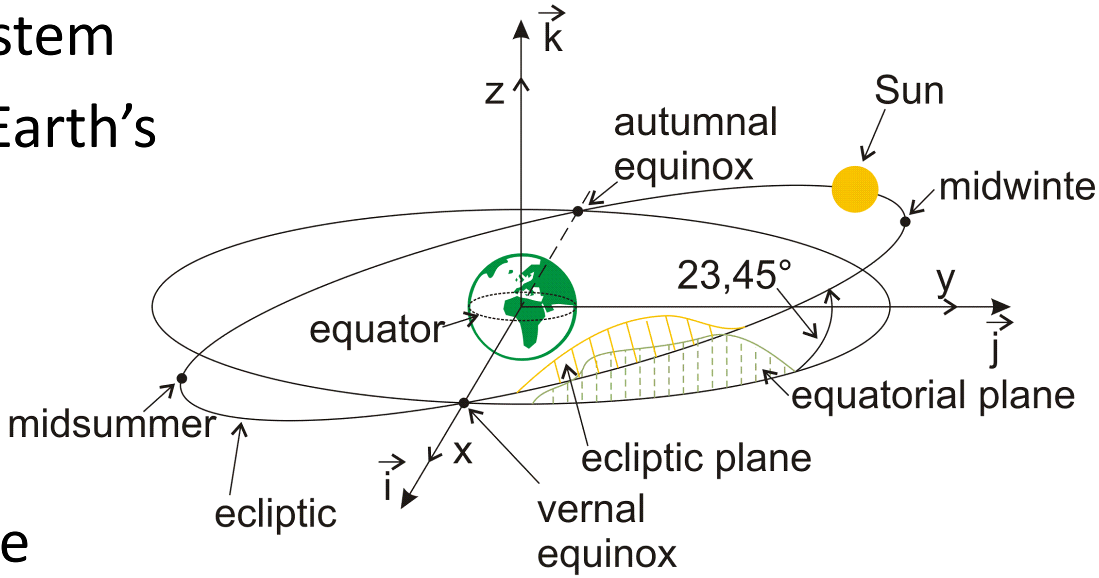
Frame of reference

- coord. system is defined by:
  - origin, fundamental plane and preferred direction
- choice of frame of reference and subsequently coordinate system depends on considered trajectory:
  - Interplanetary trajectory – Interplanetary systems, e.g. Heliocentric coordinate system
  - Earth orbits – Earth-based systems

2. Orbits in three dimensions

Earth-based systems

- Geocentric Equatorial System (GES) - the most common system in astrodynamics
  - the center of coord. system is at Earth's center
  - not-rotating coord. system
  - fundamental plane – Earth's equator plane
  - axis X points towards the vernal equinox
  - axis Z extends through the North Pole



2. Orbits in three dimensions

Earth-based systems

- Geocentric Equatorial System (GES):
  - is often considered as Earth-Centered Inertial system (ECI)
  - ECI frame of reference is not fixed in space:
    - gravitational forces of planets – **planetary precession**
    - gravitational forces of Moon and Sun – **luni-solar precession** with period 26,000 years
    - combined effect – general precession
    - inclination of Moon – additional torque on Earth's equatorial bulge – **nutation** with period 18,6 years
  - due to precession and nutation equinox is moving

2. Orbits in three dimensions

Earth-based systems

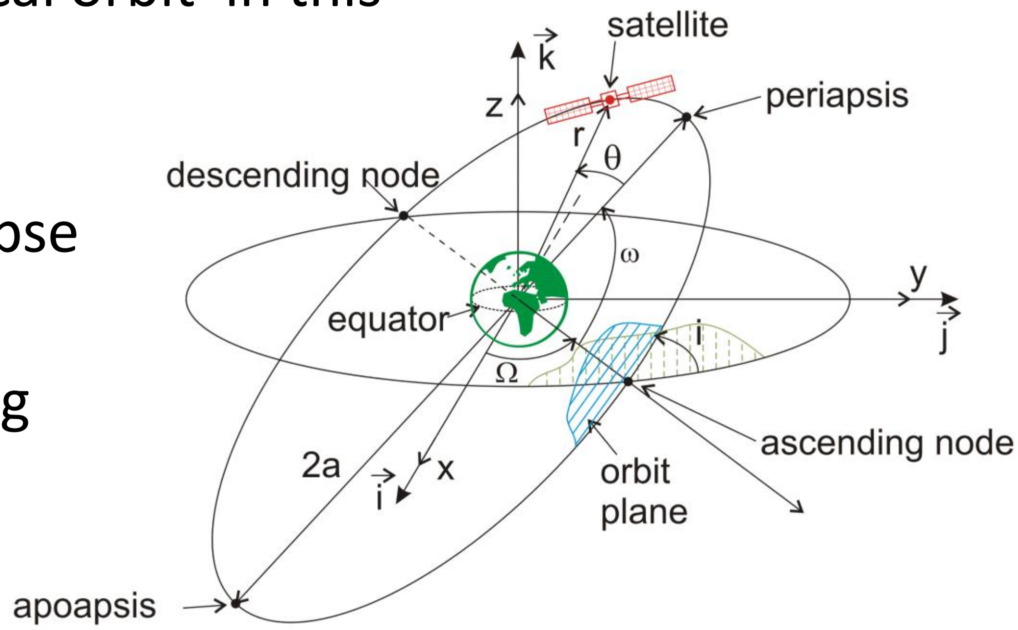
- Geocentric Equatorial System (GES):
  - for all precise applications, ECI must by defined on specific date
  - J2000 - commonly used ECI frame is defined with the Earth's Mean Equator and Equinox at 12:00 Terrestrial Time on 1 January 2000
- other Earth-based systems:
  - Earth-Centered, Earth-Fixed Coord. System – rotate with Earth
  - Perifocal Coord. System

2. Orbits in three dimensions

Orbital elements

- Location of the satellite:

1. the location of the orbital plane in defined coord. system of chosen frame of reference
   ➡ Ω, i
2. the position of the elliptical orbit in this plane
   ➡ ω
3. the characteristics of ellipse
   ➡ $e, h \text{ (or } a)$
4. the position of the moving satellite on the orbit
   ➡ $\theta \text{ (or } M)$



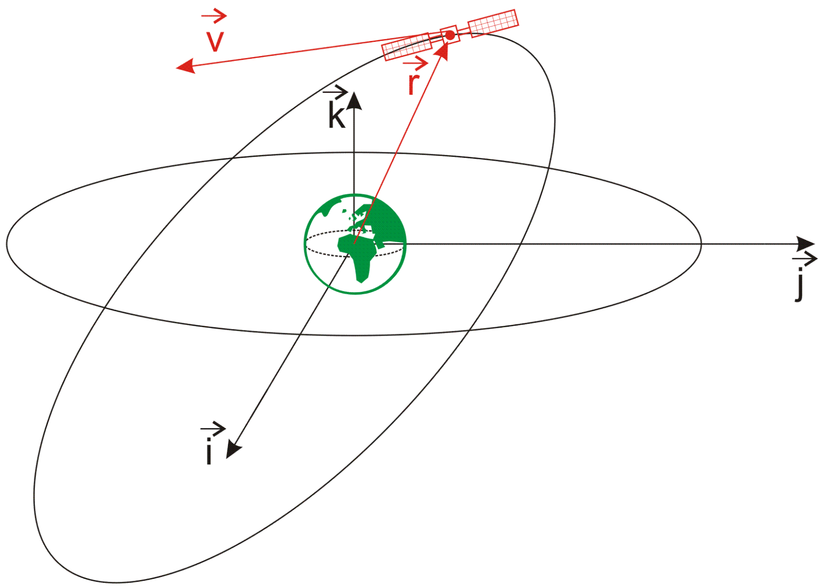
2. Orbits in three dimensions

Calculation of elements

- the goal is to determine orbital elements from:
  - position vector $\vec{r} = r_x\vec{i} + r_y\vec{j} + r_z\vec{k}$
  - velocity vector $\vec{v} = v_x\vec{i} + v_y\vec{j} + v_z\vec{k}$
  - both vectors are defined in GES at time t_0

2. Orbits in three dimensions

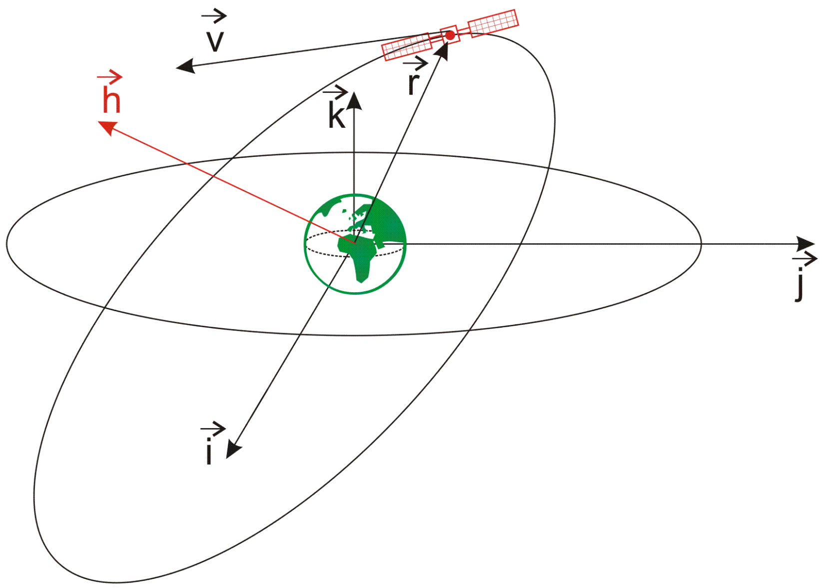
Calculation of elements



$$\vec{r} = r_x\vec{i} + r_y\vec{j} + r_z\vec{k}$$
$$\vec{v} = v_x\vec{i} + v_y\vec{j} + v_z\vec{k}$$

state vector $\vec{r}$ and $\vec{v}$

2. Orbits in three dimensions



Calculation of elements

$$\vec{r} = r_x\vec{i} + r_y\vec{j} + r_z\vec{k}$$

$$\vec{v} = v_x\vec{i} + v_y\vec{j} + v_z\vec{k}$$

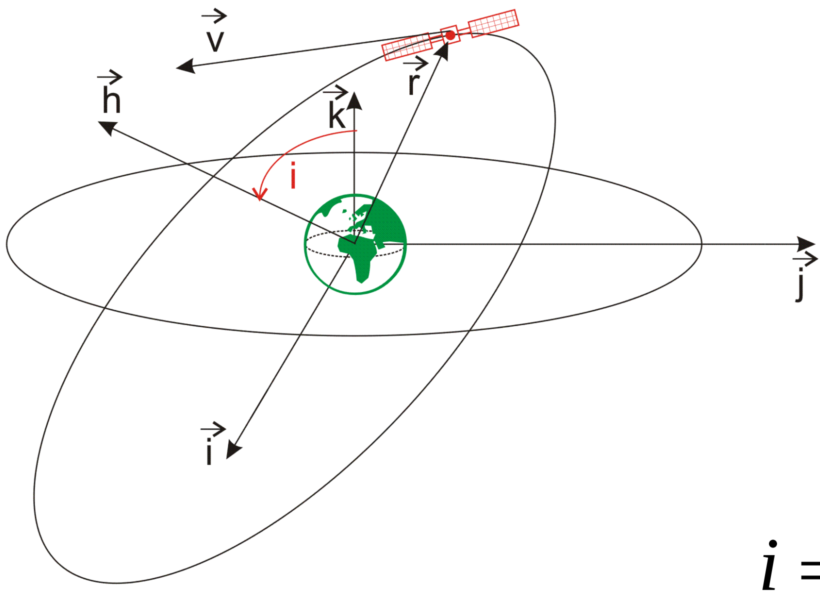
$$\vec{h} = \vec{r} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ r_x & r_y & r_z \\ v_x & v_y & v_z \end{vmatrix}$$

state vector $\vec{r}$ and $\vec{v}$

1st element $\vec{h}$

2. Orbits in three dimensions

Calculation of elements



$$\vec{r} = r_x \vec{i} + r_y \vec{j} + r_z \vec{k}$$

$$\vec{v} = v_x \vec{i} + v_y \vec{j} + v_z \vec{k}$$

$$\vec{h} = h_x \vec{i} + h_y \vec{j} + h_z \vec{k}$$

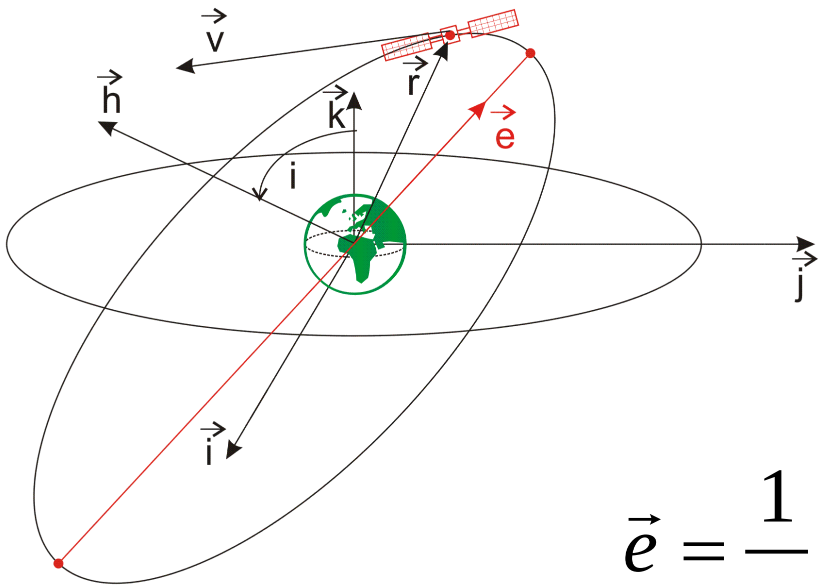
$$i = \cos^{-1}\left(\frac{h_z}{h}\right)$$

state vector $\vec{r}$ and $\vec{v}$

1st element $\vec{h}$

2nd element i

2. Orbits in three dimensions



Calculation of elements

$$\vec{r} = r_x\vec{i} + r_y\vec{j} + r_z\vec{k}$$

$$\vec{v} = v_x\vec{i} + v_y\vec{j} + v_z\vec{k}$$

$$\vec{h} = h_x\vec{i} + h_y\vec{j} + h_z\vec{k}$$

$$\vec{e} = \frac{1}{\mu}\left[\left(v^2 - \frac{\mu}{r}\right)\vec{r} - (\vec{r}\cdot\vec{v})\vec{v}\right]$$

state vector $\vec{r}$ and $\vec{v}$

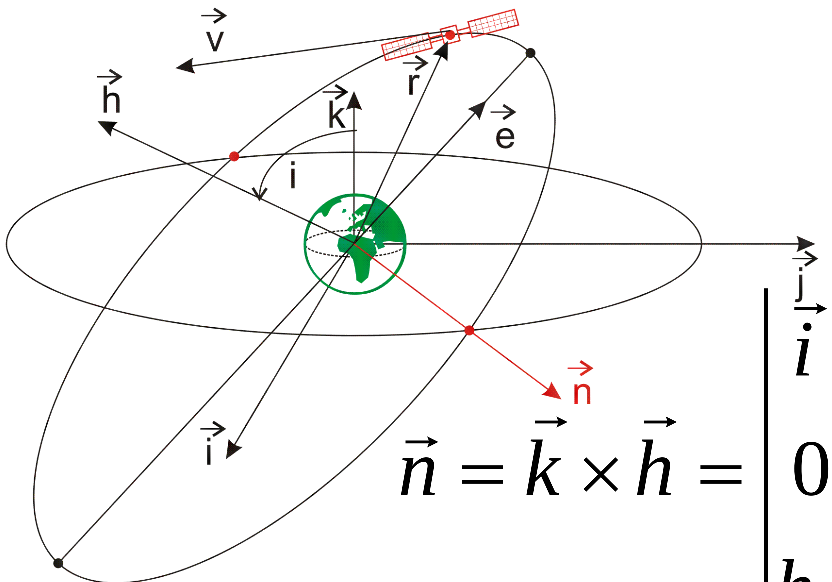
3th element $\vec{e}$

1st element $\vec{h}$

2nd element i

2. Orbits in three dimensions

Calculation of elements



$$\vec{r} = r_x\vec{i} + r_y\vec{j} + r_z\vec{k}$$

$$\vec{v} = v_x\vec{i} + v_y\vec{j} + v_z\vec{k}$$

$$\vec{h} = h_x\vec{i} + h_y\vec{j} + h_z\vec{k}$$

$$\vec{e} = e_x\vec{i} + e_y\vec{j} + e_z\vec{k}$$

$$\vec{n} = \vec{k} \times \vec{h} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & 1 \\ h_x & h_y & h_z \end{vmatrix}$$

state vector $\vec{r}$ and $\vec{v}$

3th element $\vec{e}$

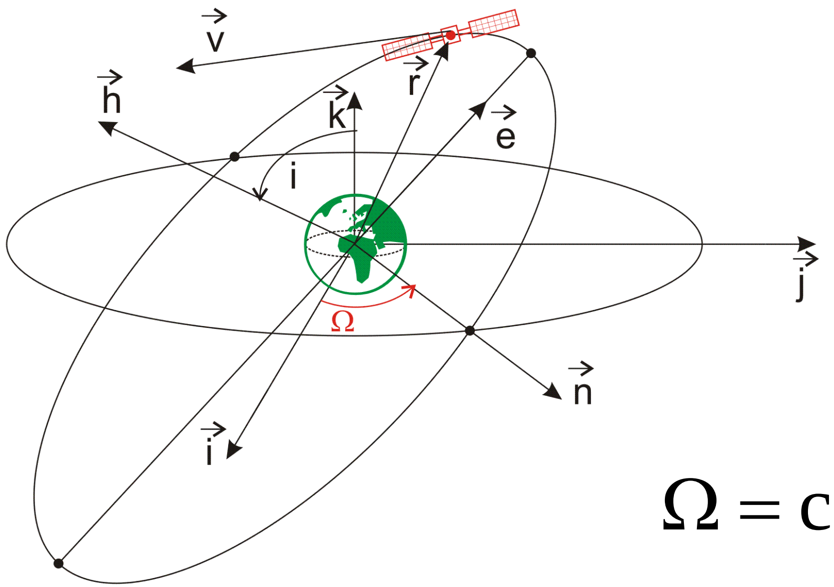
1st element $\vec{h}$

2nd element i

$\vec{n}$ **vector of node line**

2. Orbits in three dimensions

Calculation of elements



$$\vec{r} = r_x\vec{i} + r_y\vec{j} + r_z\vec{k}$$

$$\vec{v} = v_x\vec{i} + v_y\vec{j} + v_z\vec{k}$$

$$\vec{h} = h_x\vec{i} + h_y\vec{j} + h_z\vec{k}$$

$$\vec{e} = e_x\vec{i} + e_y\vec{j} + e_z\vec{k}$$

$$\vec{n} = n_x\vec{i} + n_y\vec{j} + 0\vec{k}$$

$$\Omega = \cos^{-1}\left(\frac{n_x}{n}\right)$$

3th element $\vec{e}$

state vector $\vec{r}$ and $\vec{v}$

1st element $\vec{h}$

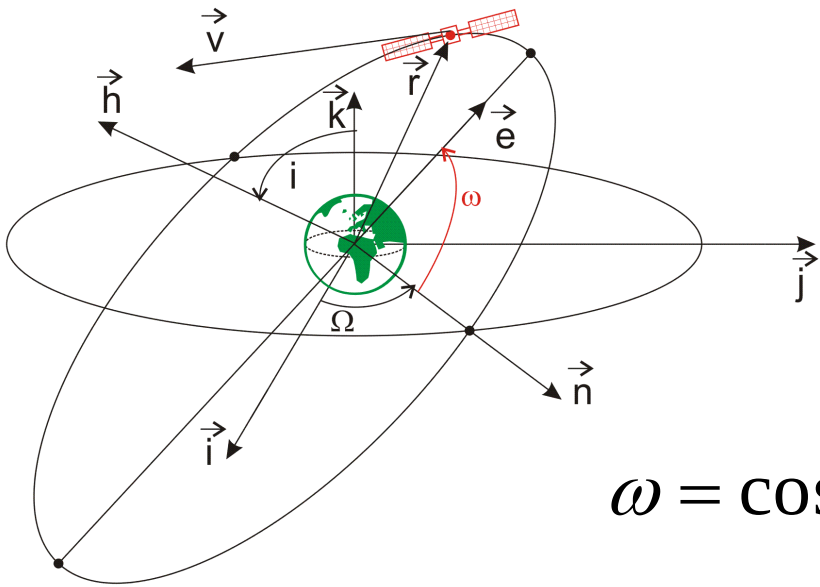
2nd element i

$\vec{n}$

Ω **4th element**

2. Orbits in three dimensions

Calculation of elements



$$\vec{r} = r_x\vec{i} + r_y\vec{j} + r_z\vec{k}$$

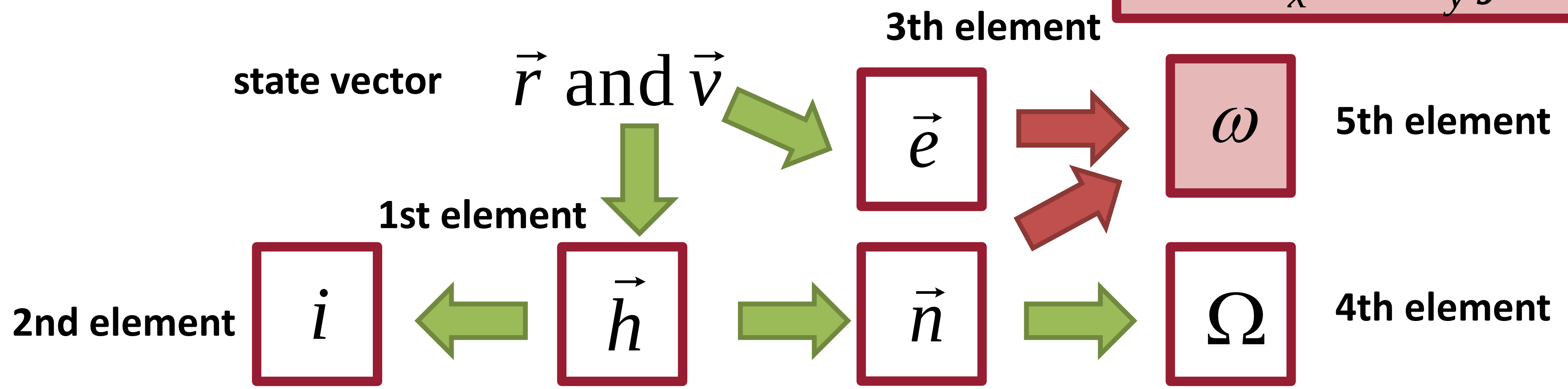
$$\vec{v} = v_x\vec{i} + v_y\vec{j} + v_z\vec{k}$$

$$\vec{h} = h_x\vec{i} + h_y\vec{j} + h_z\vec{k}$$

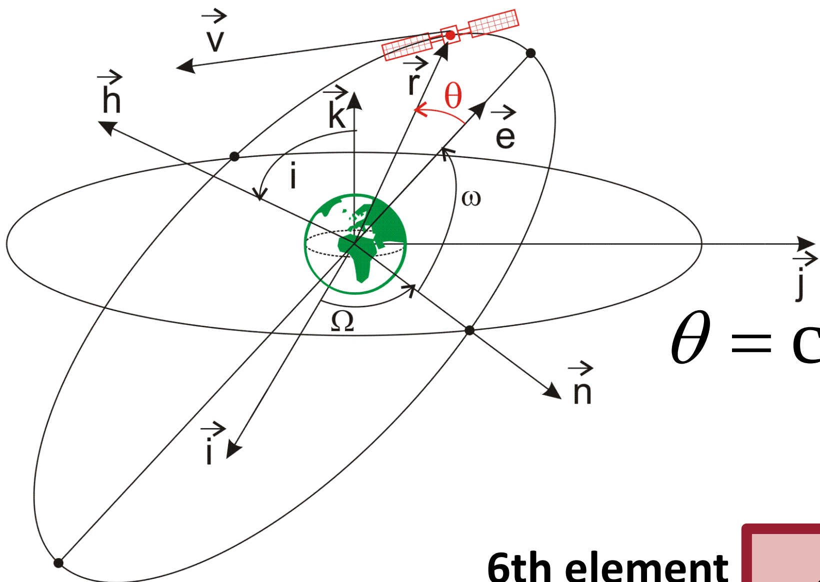
$$\vec{e} = e_x\vec{i} + e_y\vec{j} + e_z\vec{k}$$

$$\vec{n} = n_x\vec{i} + n_y\vec{j} + 0\vec{k}$$

$$\omega = \cos^{-1}\left(\frac{\vec{n}}{n} \cdot \frac{\vec{e}}{e}\right)$$



2. Orbits in three dimensions



Calculation of elements

$$\vec{r} = r_x\vec{i} + r_y\vec{j} + r_z\vec{k}$$

$$\vec{v} = v_x\vec{i} + v_y\vec{j} + v_z\vec{k}$$

$$\vec{h} = h_x\vec{i} + h_y\vec{j} + h_z\vec{k}$$

$$\vec{e} = e_x\vec{i} + e_y\vec{j} + e_z\vec{k}$$

$$\vec{n} = n_x\vec{i} + n_y\vec{j} + 0\vec{k}$$

$$\theta = \cos^{-1}\left(\frac{\vec{e}}{e} \cdot \frac{\vec{r}}{r}\right)$$

state vector $\vec{r}$ and $\vec{v}$

6th element θ

3th element $\vec{e}$

5th element ω

1st element $\vec{h}$

2nd element i

$\vec{n}$

4th element Ω

2. Orbits in three dimensions

Calculation of elements

Example:

input parameters:

$$r_p = \left[2004.75,\ 6174.08,\ 1567.56\right] \text{km}$$

$$v_p = \left[-7.556,\ 1.581,\ 3.435\right] \text{km/s}$$

2. Orbits in three dimensions

Calculation of elements

Example:

$$h = 56430.1 \ \text{km}^2/\text{s}$$

$$i = 28°$$

$$e = 0.196$$

$$\Omega = 45°$$

$$\omega = 30°$$

$$\theta = 0°$$

input parameters:

$$r_p = [2004.75, \ 6174.08, \ 1567.56] \text{km}$$

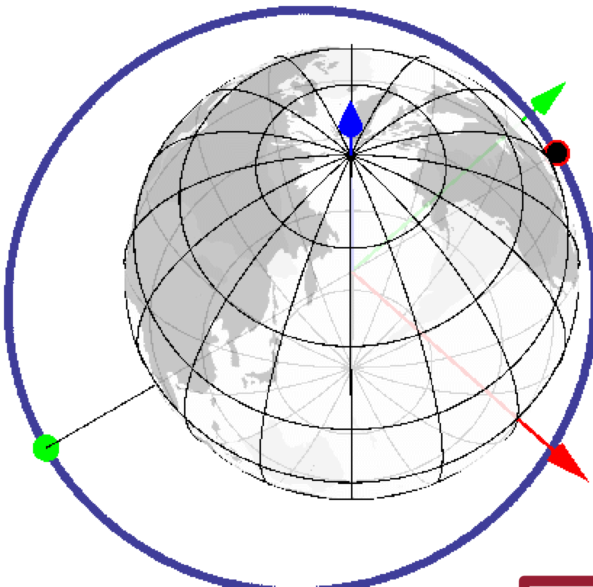
$$v_p = [-7.556, \ 1.581, \ 3.435] \text{km/s}$$

2. Orbits in three dimensions

Calculation of elements

Example:

Orbit in 2D view:



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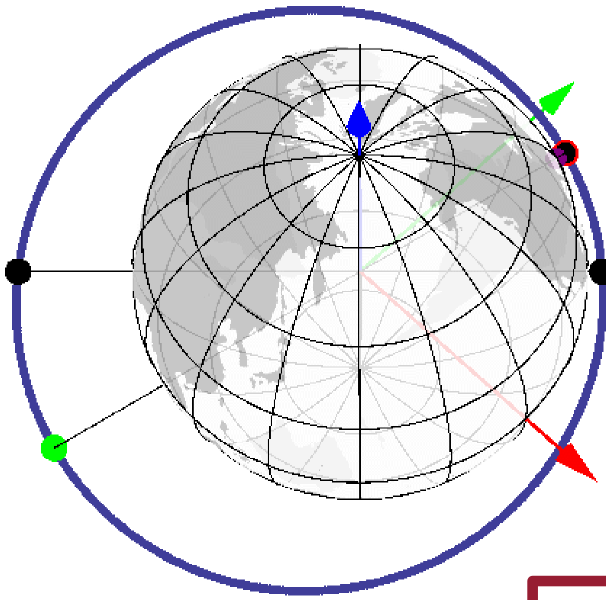
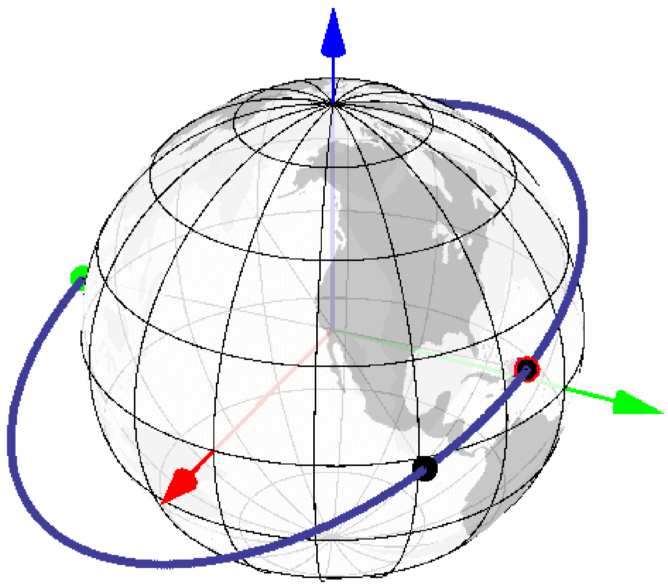
2. Orbits in three dimensions

Calculation of elements

Example:

Orbit in 3D view:





input parameters:

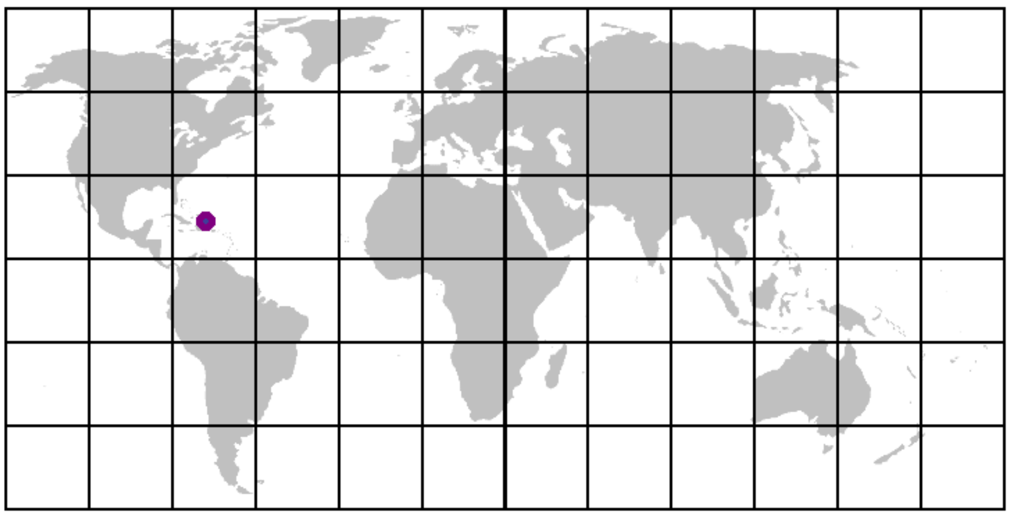
$$r_p = [2004.75,\ 6174.08,\ 1567.56\ ]\,\text{km}$$
$$v_p = [-7.556,\ 1.581,\ 3.435\ ]\,\text{km/s}$$

2. Orbits in three dimensions

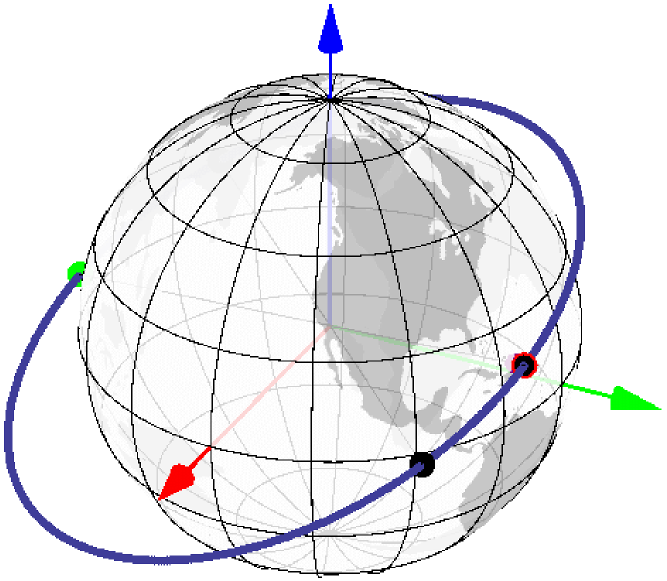
Calculation of elements

Example:

Orbit in 2D map:



Orbit in 3D view:



input parameters:

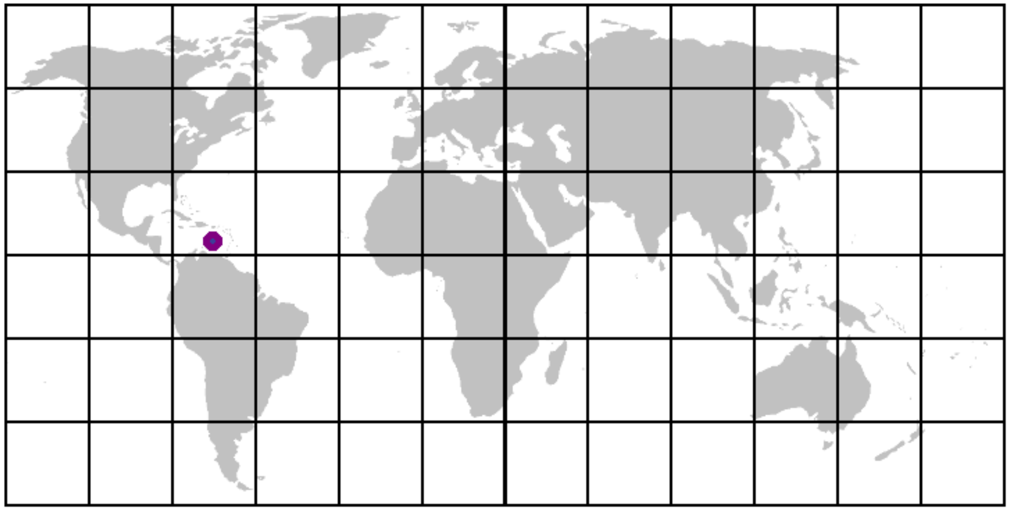
$$r_p = [2004.75,\ 6174.08,\ 1567.56\ ]\,\text{km}$$
$$v_p = [-7.556,\ 1.581,\ 3.435\ ]\,\text{km/s}$$

2. Orbits in three dimensions

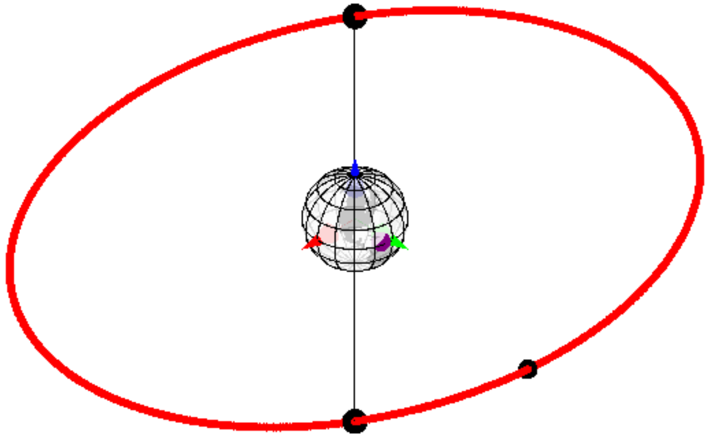
Calculation of elements

Example: GEO

Orbit in 2D map:



Orbit in 3D view:



input parameters:

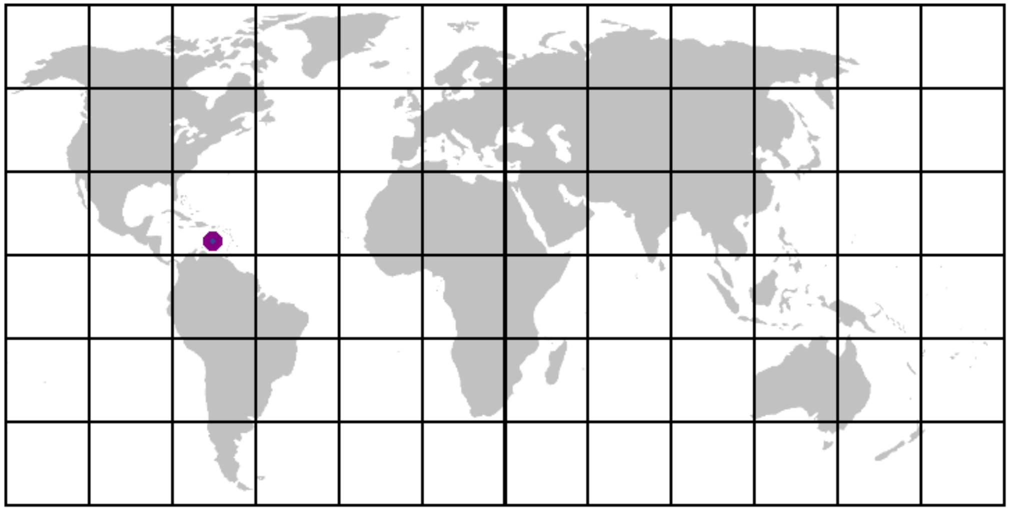
GEO, circular orbit,
$i = 2.5°$

2. Orbits in three dimensions

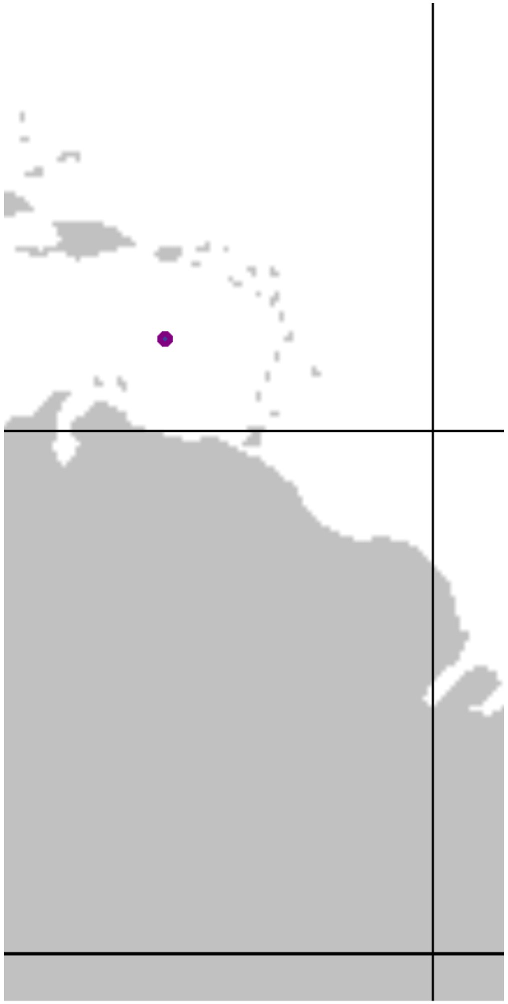
Calculation of elements

Example: GEO

Orbit in 2D map:



Detail view:



input parameters:

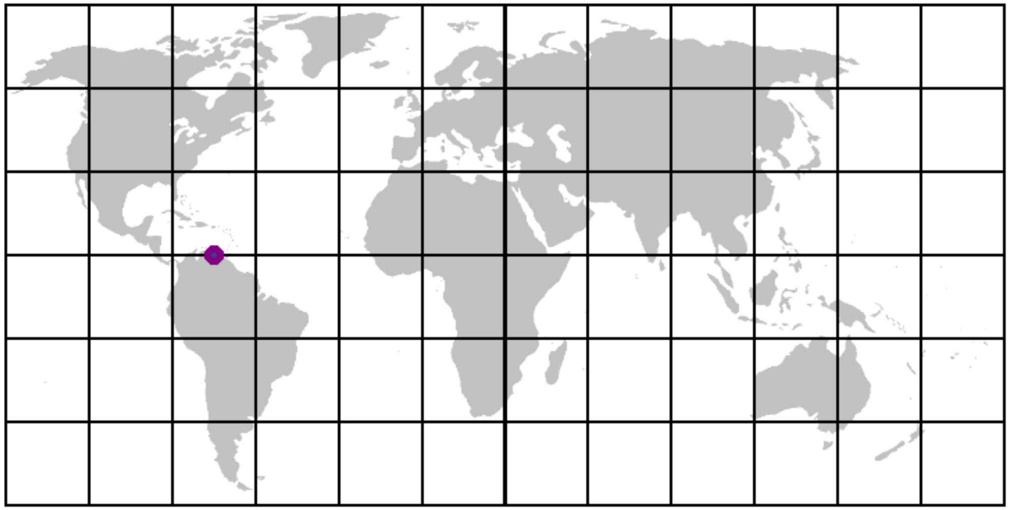
GEO, circular orbit,
$i = 2.5°$

2. Orbits in three dimensions

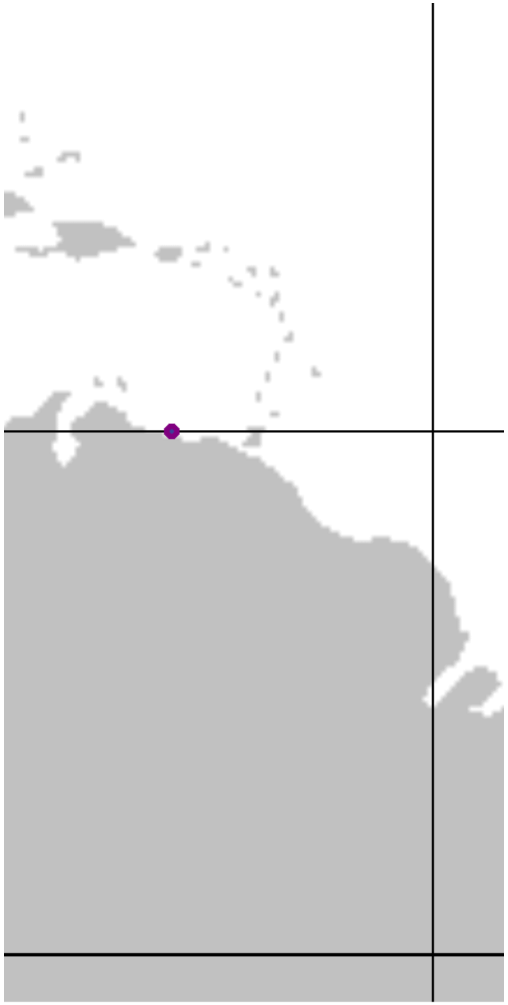
Calculation of elements

Example: GEO

Orbit in 2D map:



Detail view:



input parameters:

GEO, $e = 0.01575$

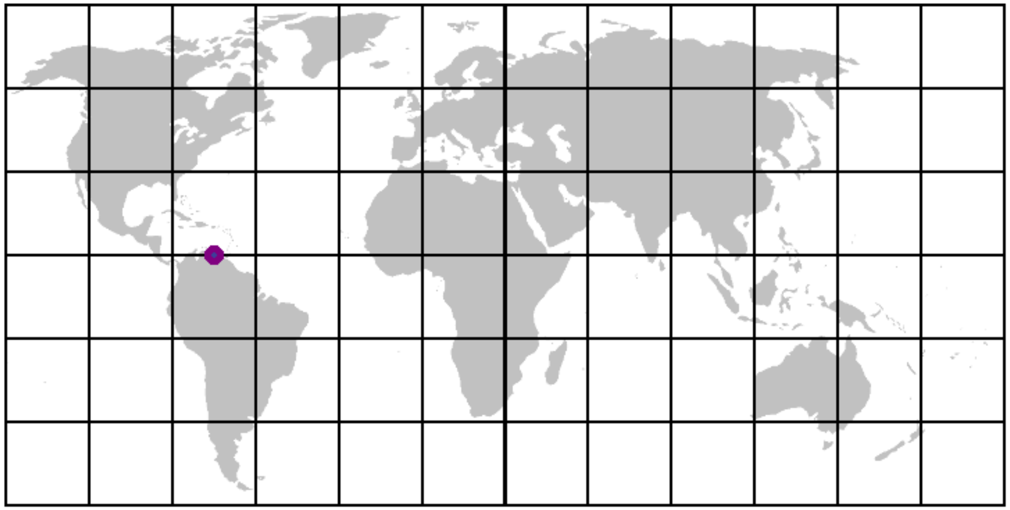
$i = 0°$

2. Orbits in three dimensions

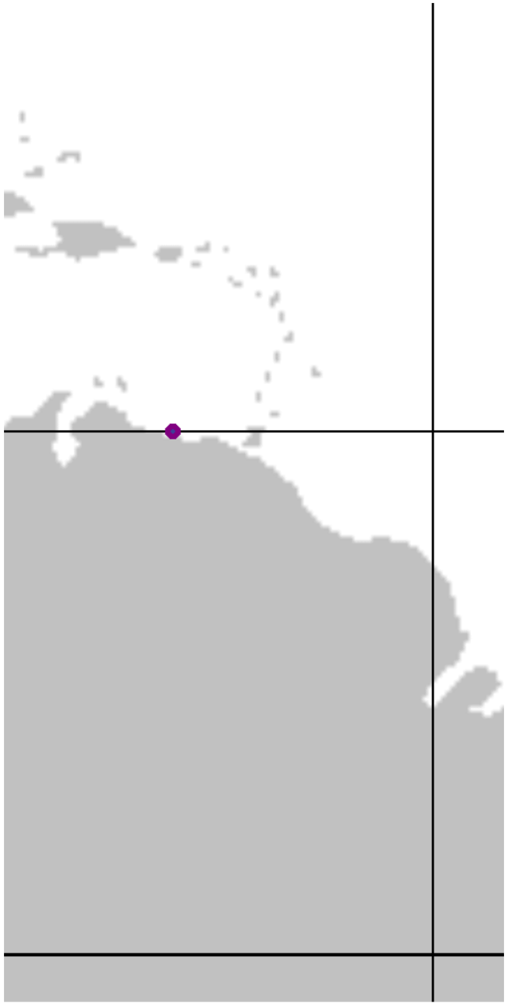
Calculation of elements

Example: GEO

Orbit in 2D map:



Detail view:



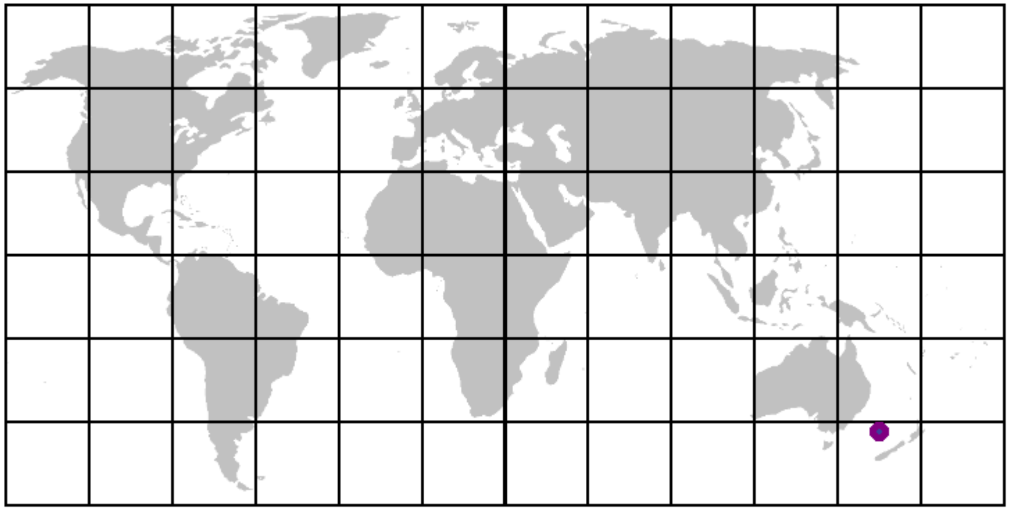
input parameters:

GEO, $e = 0.01575$
$i = 2.5°$

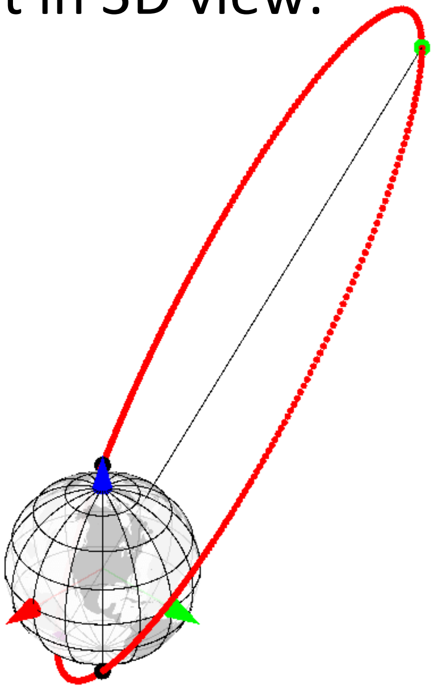
2. Orbits in three dimensions

Example: Molnija

Orbit in 2D map:



Orbit in 3D view:



input parameters:

$$e = 0.75 \quad h_a = 40089 \text{ km}$$
$$i = 63.41^\circ \quad h_p = 260 \text{ km}$$

3. Orbital perturbations

- Perturbing forces
- Geopotential
- Orbit propagation
- Variation of parameters
- Examples of orbits

3. Orbital perturbations

Perturbing forces

Orbits of Earth satellites are influenced by 2 facts:

- The Earth is not exactly spherical and the mass distribution is not exactly spherically symmetric
- The satellite feels other forces apart from the Earth's attraction:
  - attractive forces due to other heavenly bodies
  - forces that can be globally categorized as frictional

All these influences are called perturbations

3. Orbital perturbations

Perturbing forces

Conservative forces – can be derived from potential:

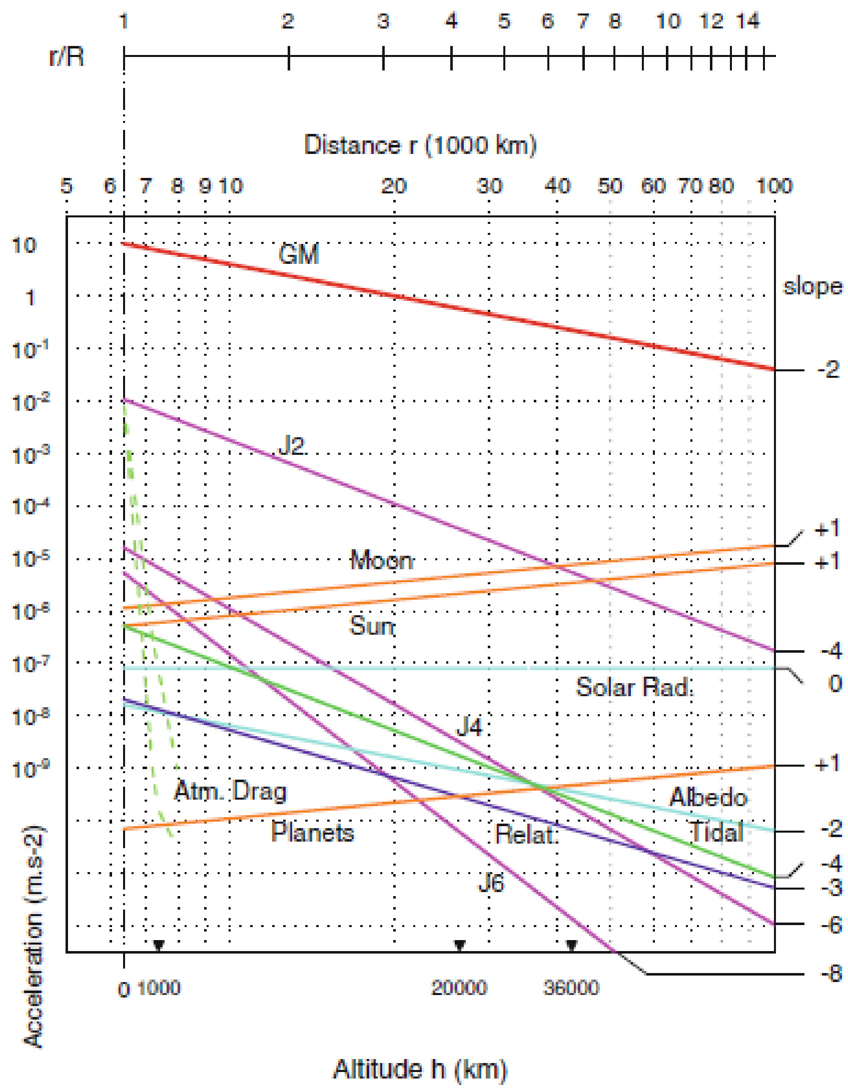
- flattening of the Earth
- Attraction of the Moon
- Attraction of the Sun
- Attraction by other planets

Non-conservative forces – cannot be derived from potential – dissipative forces:

- atmospheric drag
- radiation pressure

3. Orbital perturbations

Perturbing forces



source: Capderou: Handbook of Satellite Orbits

Influence of perturbing forces expressed by accelerations:

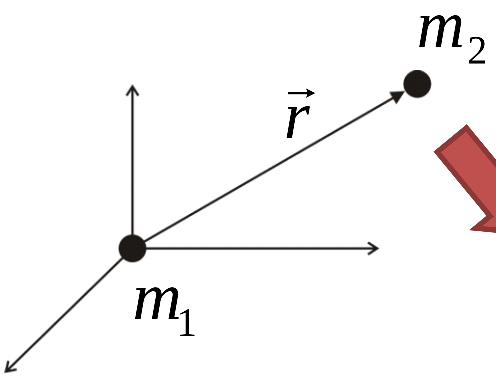
- GM – attraction of Earth (sphere shape)
- J2 – flattening of the Earth (Earth ellipsoid)
- J4, J6 – potential of Earth expressed by higher orders
- Moon, Sun, Planets – their attraction

3. Orbital perturbations

Geopotential

Potential of single mass point:

position of masses



potential energy of mass m_2 in gravitational field of mass m_1

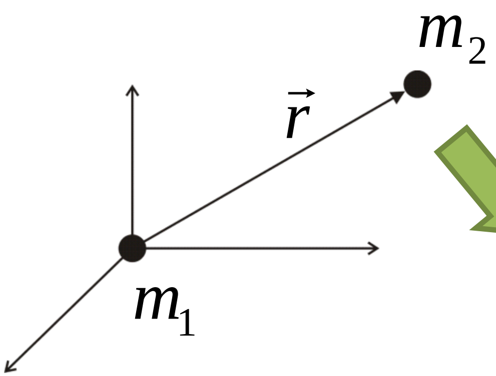
$$E_p = -G\frac{m_1 m_2}{r}$$

3. Orbital perturbations

Geopotential

Potential of single mass point:

position of masses



potential energy of mass m_2 in gravitational field of mass m_1

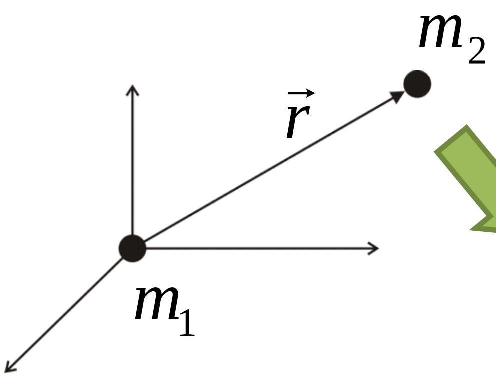
$$E_p = -G\frac{m_1 m_2}{r}$$

gravitational potential

$$U(r) = -\frac{E_p}{m_2} = \frac{\mu}{r}$$

Potential of single mass point:

position of masses



potential energy of mass m_2 in gravitational field of mass m_1

$$E_p = -G\frac{m_1 m_2}{r}$$

gravitational potential

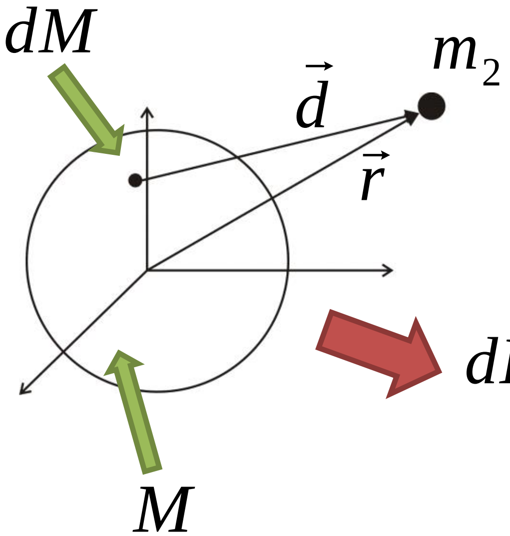
$$U(r) = -\frac{E_p}{m_2} = \frac{\mu}{r}$$

equation of motion expressed by potential

$$\ddot{\vec{r}} = \operatorname{grad}(U)$$

Potential of Earth: 1. approximation - sphere

position of masses



potential energy of mass m_2 in gravitational field of dM

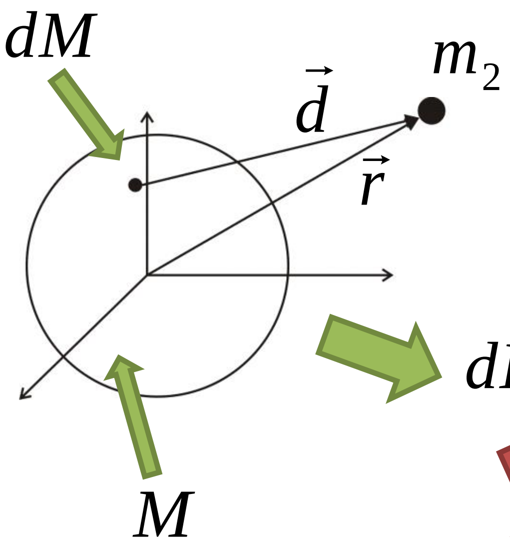
$$dE_p = -G\frac{dMm_2}{d}$$

3. Orbital perturbations

Geopotential

Potential of Earth: 1. approximation - sphere

position of masses



potential energy of mass m_2 in gravitational field of dM

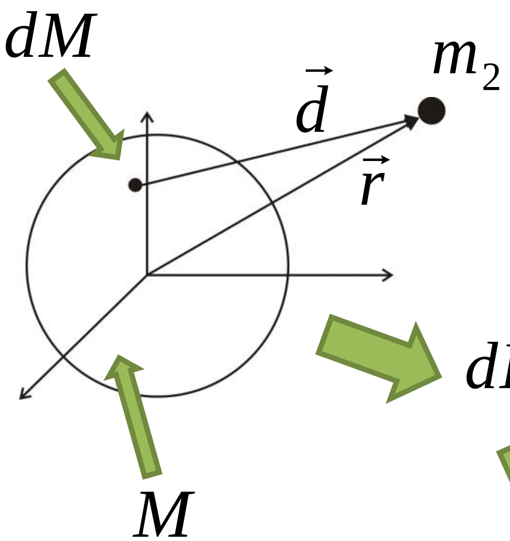
$$dE_p = -G\frac{dMm_2}{d}$$

gravitational potential

$$dU = -\frac{dE_p}{m_2} = \frac{GdM}{d}$$

Potential of Earth: 1. approximation - sphere

position of masses



potential energy of mass m_2 in gravitational field of dM

$$dE_p = -G\frac{dMm_2}{d}$$

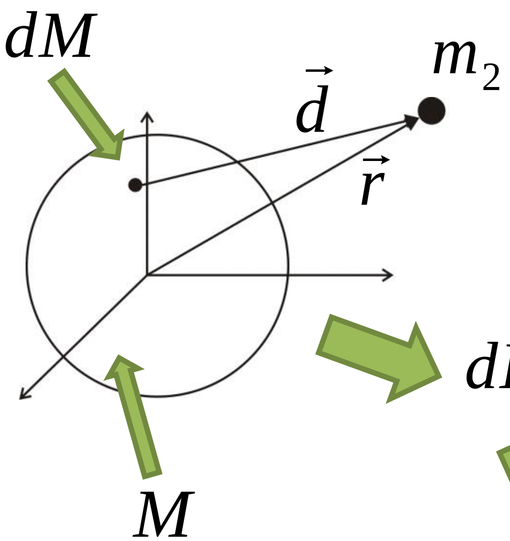
gravitational potential

$$dU = -\frac{dE_p}{m_2} = \frac{GdM}{d}$$

$$U = \int_M dU = \int_M \frac{GdM}{d}$$

Potential of Earth: 1. approximation - sphere

position of masses



potential energy of mass m_2 in gravitational field of dM

$$dE_p = -G\frac{dMm_2}{d}$$

gravitational potential

$$dU = -\frac{dE_p}{m_2} = \frac{GdM}{d}$$

$$U = \int_M dU = \int_M \frac{GdM}{d}$$

integration over sphere boundary

equal potential as single mass potential

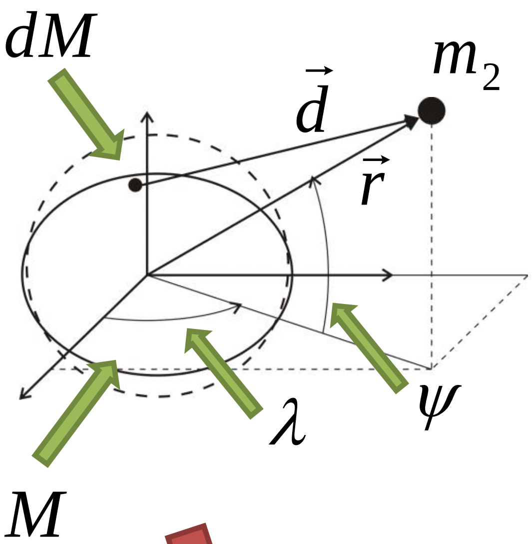
$$U(r) = \frac{GM}{r} = \frac{\mu}{r}$$

3. Orbital perturbations

Geopotential

Potential of Earth: 2. approximation - ellipsoid

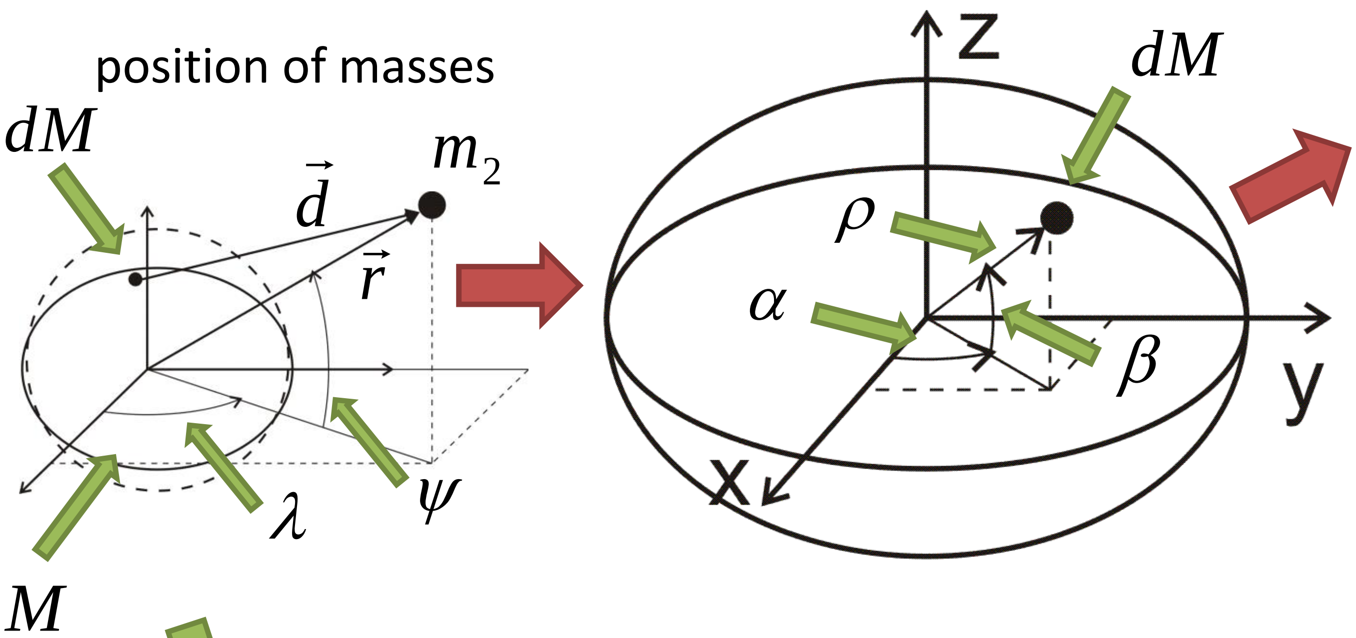
position of masses



Position of m_2:

- longitude λ
- latitude ψ
- radius r

Potential of Earth: 2. approximation - ellipsoid



Position of m_2:
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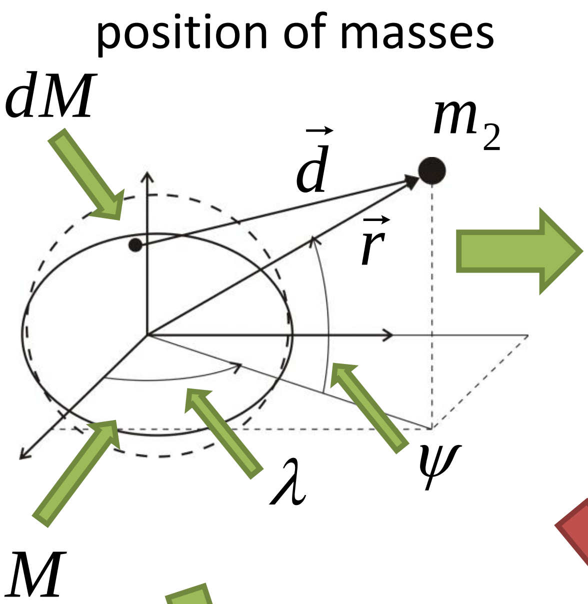
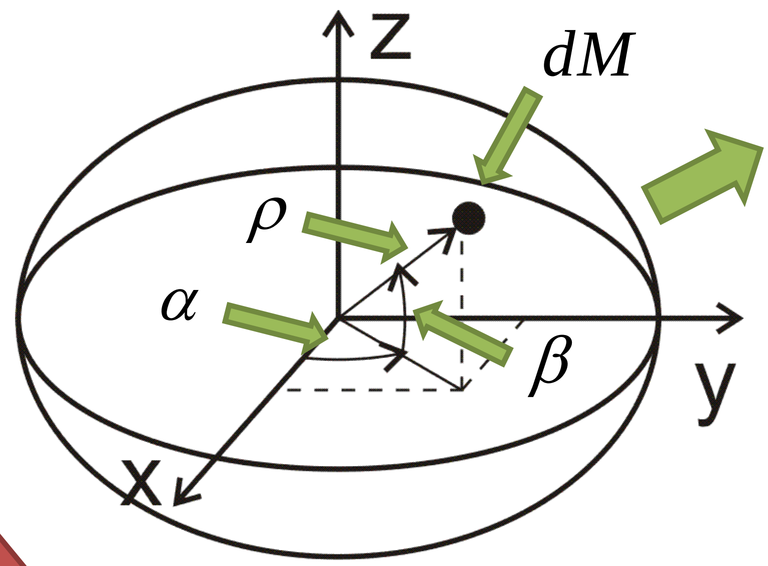
Ellipsoid:
- longitude α
- latitude β
- radius ρ

Potential of Earth: 2. approximation - ellipsoid





Ellipsoid:

- longitude α
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Position of m_2:

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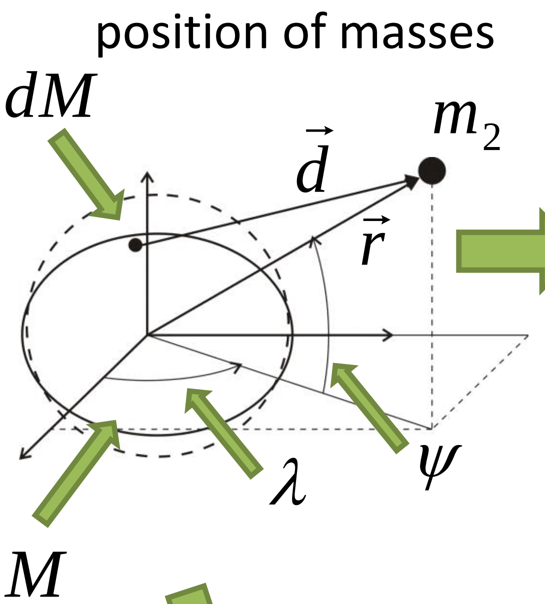
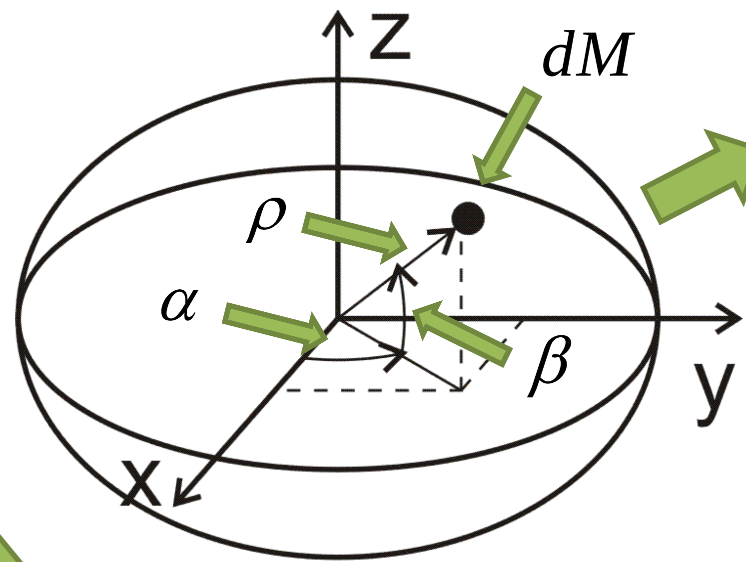
- latitude ψ
- radius r

σ angle between $\vec{r}$ and $\vec{d}$

$$d = r\sqrt{1-2\frac{\rho}{r}\cos\sigma+\left(\frac{\rho}{r}\right)^2}$$

Potential of Earth: 2. approximation - ellipsoid





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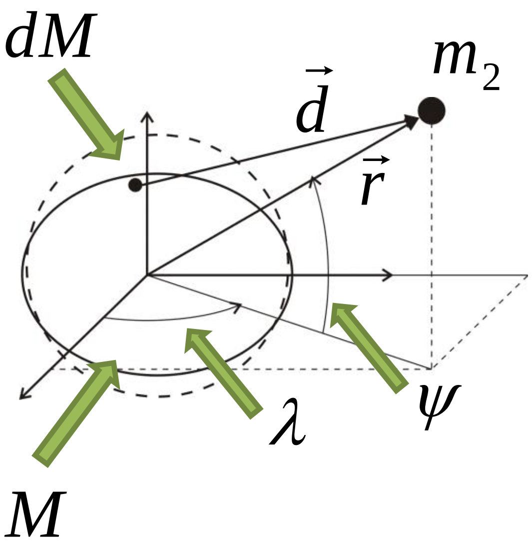
$$U = \int_M \frac{GdM}{d}$$

Potential of Earth: 2. approximation - ellipsoid



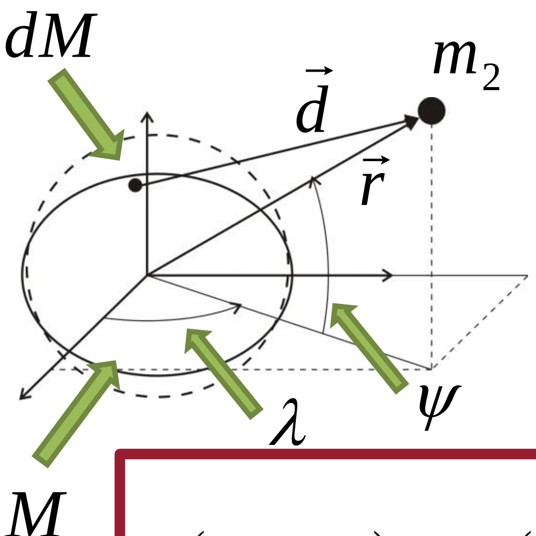
using:

- expansion of *1/d* in terms of Legendre polynomials
- symmetric properties of ellipsoid

$$U = \int_M \frac{GdM}{d}$$

Potential of Earth: 2. approximation - ellipsoid

position of masses



using:

- expansion of *1/d* in terms of Legendre polynomials
- symmetric properties of ellipsoid

$$U = \int_M \frac{G\,dM}{d}$$

$$U(r,\lambda,\psi) = U(r,\psi) = \frac{\mu}{r}\left(1 - \left(\frac{R}{r}\right)^2 J_2 \frac{3\sin^2\psi - 1}{2}\right)$$

R is equatorial radius

J_2 dimensionless coefficient

$$J_2 = \frac{1}{MR^2}\left(I_x - I_z\right)$$

$$J_2 = 1.0826 \times 10^{-3}$$

3. Orbital perturbations

Geopotential

Potential of Earth: expansion to higher degrees

- potential is function of all 3 coordinates, i.e. ⇒ $U(r, \lambda, \psi)$

3. Orbital perturbations

Geopotential

Potential of Earth: expansion to higher degrees

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$$U(r,\lambda,\psi)=\frac{\mu}{r}\sum_{l=0}^{\infty}\left(\frac{R}{r}\right)^{l}\left[\sum_{m=0}^{l}\left(C_{lm}\cos(m\lambda)+S_{lm}\sin(m\lambda)\right)P_{lm}(\sin\psi)\right]$$

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parameters are obtained from precise observation of the motion of satellites

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Legendre functions

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Legendre functions

products

$\sin(m\lambda)P_{lm}(\sin\psi)$

$\cos(m\lambda)P_{lm}(\sin\psi)$

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$$\sin(m\lambda)P_{lm}(\sin\psi)$$

$$\cos(m\lambda)P_{lm}(\sin\psi)$$

$$H_{lm}(\lambda,\psi)=P_{lm}(\sin\psi)\mathrm{e}^{im\lambda}$$

Complex functions called Spherical Harmonics

3. Orbital perturbations

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$$U(r,\lambda,\psi)=\frac{\mu}{r}\sum_{l=0}^{\infty}\left(\frac{R}{r}\right)^{l}\left[\sum_{m=0}^{l}\left(C_{lm}\cos(m\lambda)+S_{lm}\sin(m\lambda)\right)P_{lm}(\sin\psi)\right]$$

- for m=0: $C_{l0}=J_l$, $S_{l0}=0$, $H_{l0}(\lambda,\psi)$ are called zonal harmonics,
- for m=l: $H_{ll}(\lambda,\psi)$ are called sectoral harmonics
- all other functions $H_{lm}(\lambda,\psi)$ are called tesseral harmonics

3. Orbital perturbations

Geopotential

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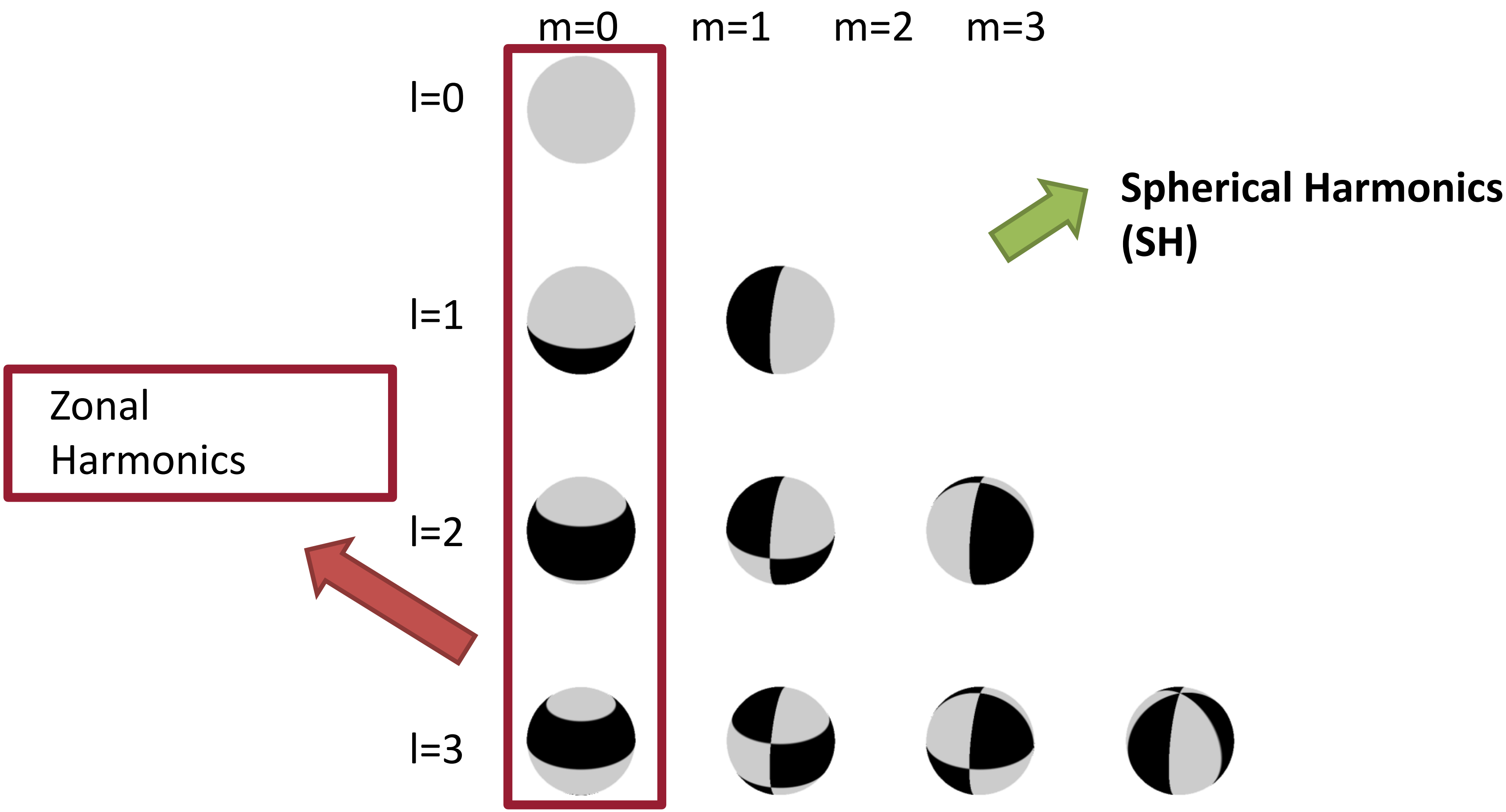

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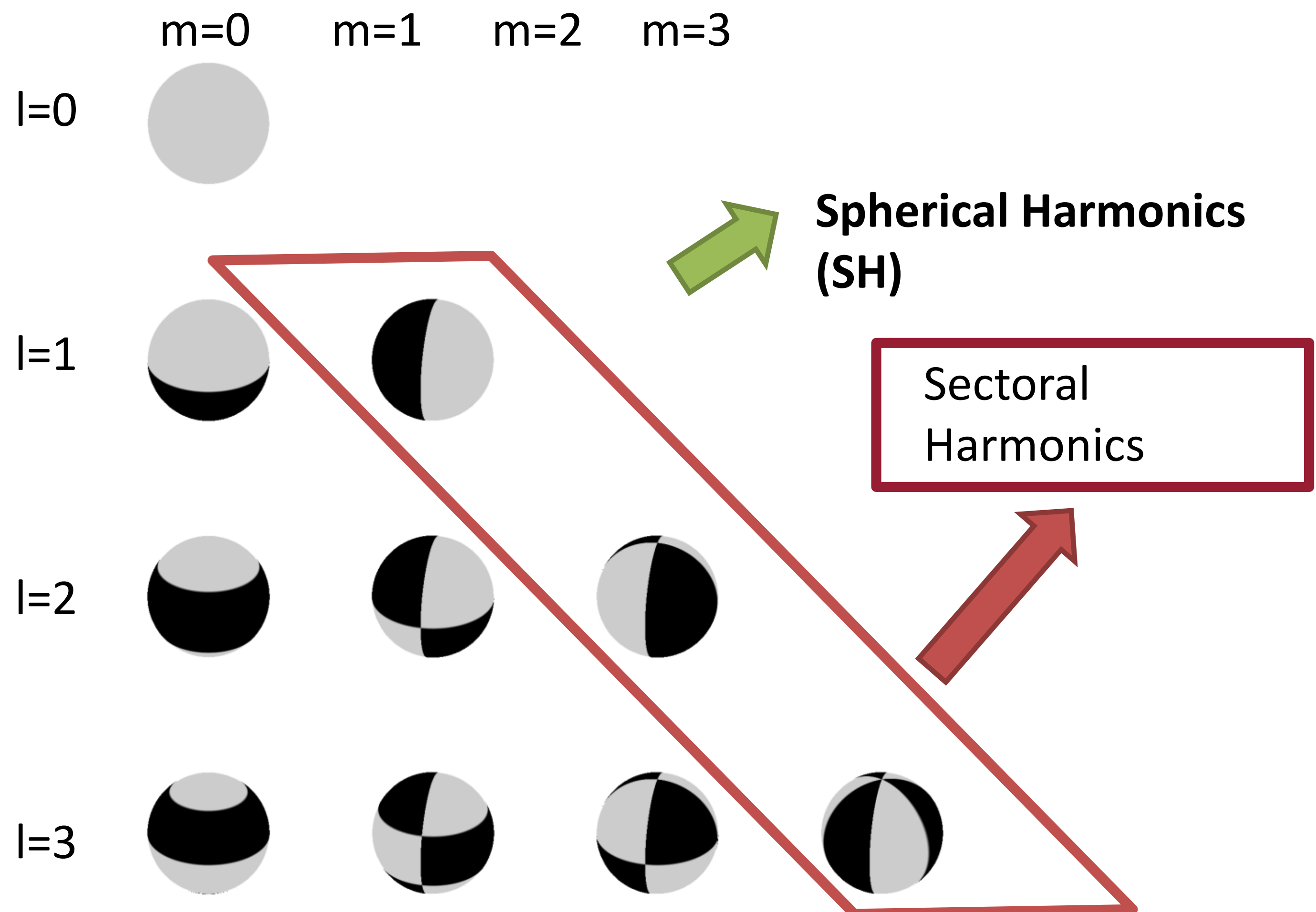

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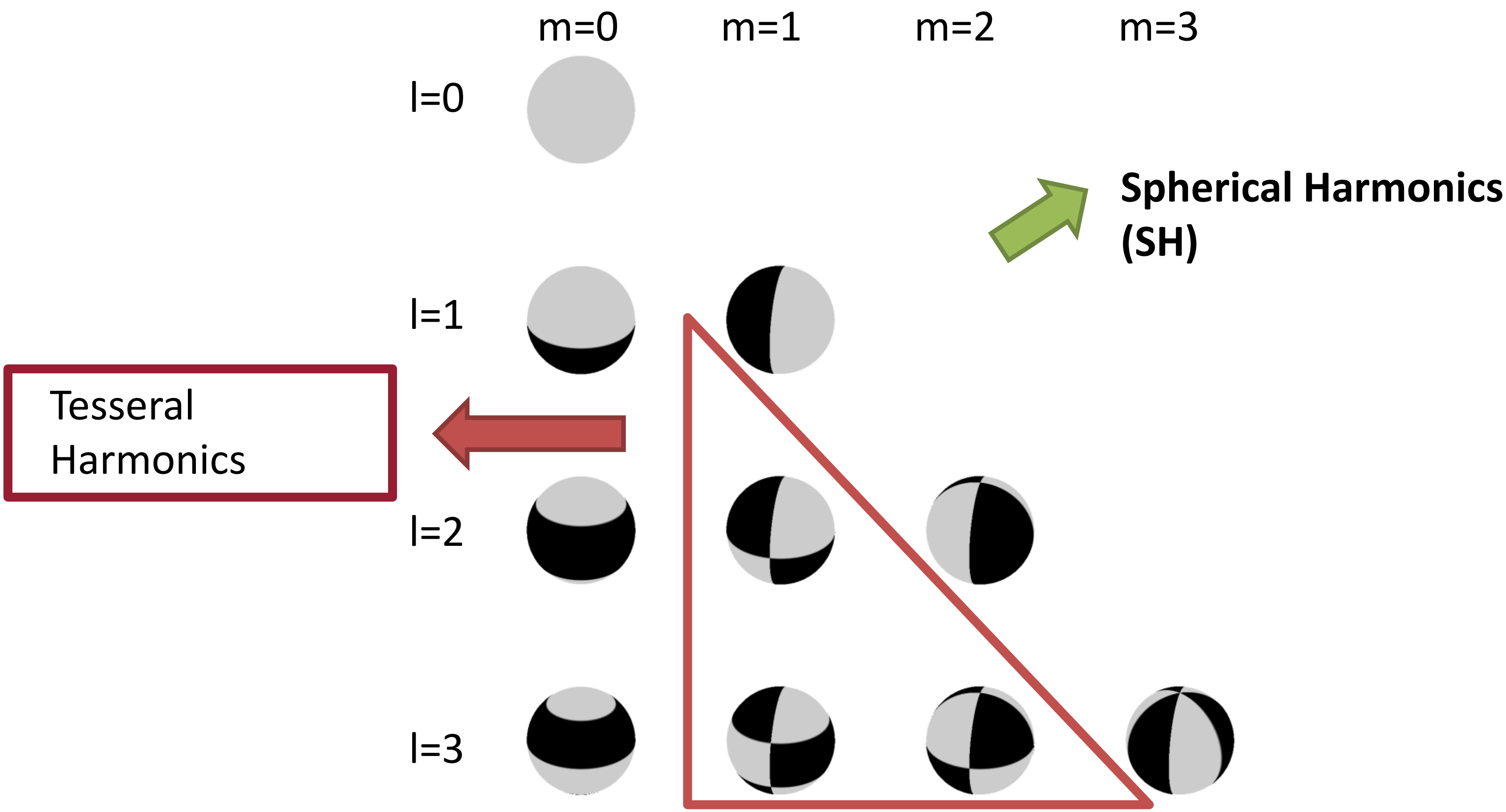

3. Orbital perturbations

Geopotential

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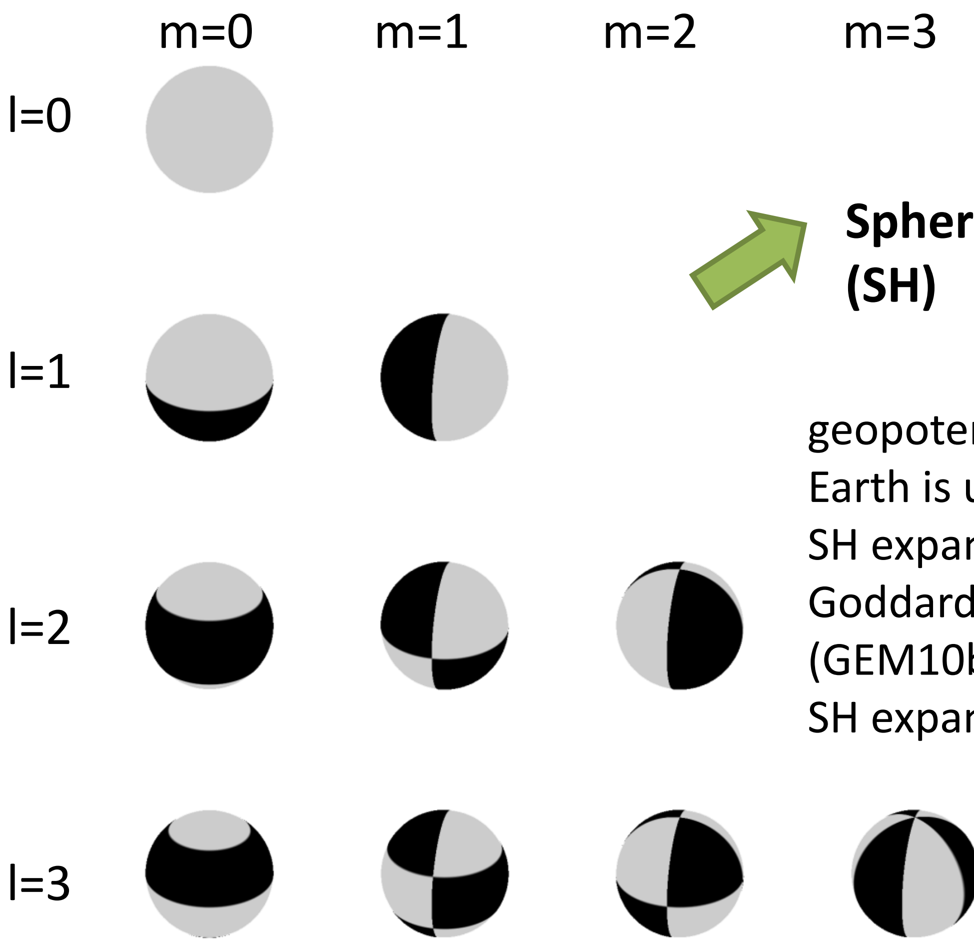

Spherical Harmonics (SH)

geopotential model of Earth is using coefficients in SH expansion, for example Goddard Earth Model 10b (GEM10b) is using 21x21 SH expansion

3. Orbital perturbations

Orbit propagation

- the goal is to solve equation of motion with initial conditions

$$\ddot{\vec{r}} = \operatorname{grad}(U)$$

$$\vec{r}(t=0) = \vec{r}_0 \quad \text{and} \quad \dot{\vec{r}}(t=0) = \dot{\vec{r}}_0$$

- potential *U* expresses influence of central acceleration and perturbative acceleration

$$U = U_0 + R$$

- for example, perturbative potential

$$U_0 = \frac{\mu}{r} \qquad\qquad R = -\frac{\mu R^2}{r^3} J_2 \frac{3\sin^2\psi - 1}{2}$$

3. Orbital perturbations

Orbit propagation

- analytical methods: ➡ **general perturbations**
  - expresses modification of motion
  - enable to determine whether the eccentricity increases, the orbit begins to precess, and so on
- numerical methods: ➡ **special perturbations**
  - one step methods – purely mathematical approach: Runge-Kuta
  - multistep methods – methods developed by astronomers to determine the motions of planets: Adams-Bashforth, Adams-Moulton
  - special methods design specially for artificial satellites

3. Orbital perturbations

Variation of parameters

- Variation of parameters is analytical method to investigate influence of perturbation on planetary or satellite motion
- Mathematical intro:

diff. equation with
right hand side

$$\frac{dy}{dt} + f(t)y = g(t)$$

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$$\frac{dy}{dt} + f(t)y = g(t)$$

↓

$$\frac{dy}{dt} + f(t)y = 0$$

homogenous equation

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$$\frac{dy}{dt} + f(t)y = g(t)$$

$$\downarrow$$

$$\frac{dy}{dt} + f(t)y = 0 \quad \Rightarrow \quad \frac{dy}{y} = -f(t)dt$$

homogenous equation

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$$\frac{dy}{dt} + f(t)y = g(t)$$

$$\frac{dy}{dt} + f(t)y = 0 \quad \Rightarrow \quad \frac{dy}{y} = -f(t)dt \quad \Rightarrow \quad y = ce^{-\int f(t)dt}$$

homogenous equation

homogenous solution
c – int. constant

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to obtain solution of eq. with right hand side, we allow c to be function of t

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$$\frac{dc}{dt} e^{-\int f(t)dt} = g(t)$$

homogenous equation

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homogenous solution

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$$c(t) = C + \int g(t) \mathrm{e}^{\int f(t)dt} dt$$

$$\frac{dc}{dt} \mathrm{e}^{-\int f(t)dt} = g(t)$$

diff. equation with right hand side

$$\frac{dy}{dt} + f(t)y = g(t)$$

to obtain solution of eq. with right hand side, we allow c to be function of t

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homogenous equation

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homogenous solution

c – int. constant

3. Orbital perturbations

Variation of parameters

- Similar process can be applied to system of diff. eq.
- diff. equation of motion can be written as system of equations

$$\frac{d\vec{r}}{dt} = \vec{v}$$

$$\frac{d\vec{v}}{dt} + \frac{\mu}{r^3}\vec{r} = \mathrm{grad}(R)$$

3. Orbital perturbations

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$$\frac{d\vec{r}}{dt} = \vec{v}$$

$$\frac{d\vec{v}}{dt} + \frac{\mu}{r^3}\vec{r} = \text{grad}(R)$$

→ solution without right hand side →

$$\vec{r} = \vec{r}(t, 6\,\text{constants})$$

$$\vec{v} = \vec{v}(t, 6\,\text{constants})$$

3. Orbital perturbations

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↓

6 int. constants are 6 orbital elements

$$\Omega, i, \omega, a, e, M$$

3. Orbital perturbations

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↓

6 int. constants are 6 orbital elements $\Omega, i, \omega, a, e, M$

←

variation of all 6 orbital elements $\Omega(t), i(t), \omega(t), a(t), e(t), M(t)$

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↓

6 int. constants are 6 orbital elements $\Omega, i, \omega, a, e, M$

← variation of all 6 orbital elements $\Omega(t), i(t), \omega(t), a(t), e(t), M(t)$

← calculation of parameters

3. Orbital perturbations

Variation of parameters

- Similar process can be applied to system of diff. eq.
- diff. equation of motion can be written as system of equations

$$\frac{d\Omega}{dt} = \frac{1}{nab\sin i}\frac{\partial R}{\partial i}$$

$$\frac{di}{dt} = -\frac{1}{nab\sin i}\frac{\partial R}{\partial \Omega} + \frac{\cos i}{nab\sin i}\frac{\partial R}{\partial \omega}$$

$$\frac{dM}{dt} = -\frac{2}{na}\frac{\partial R}{\partial a} - \frac{b^2}{na^4 e}\frac{\partial R}{\partial e}$$

$$\frac{da}{dt} = \frac{2}{na}\frac{\partial R}{\partial M}$$

$$\frac{de}{dt} = -\frac{b}{na^3 e}\frac{\partial R}{\partial \omega} + \frac{b^2}{na^4 e}\frac{\partial R}{\partial M}$$

$$\frac{d\omega}{dt} = -\frac{\cos i}{nab\sin i}\frac{\partial R}{\partial i} + \frac{b}{na^3 e}\frac{\partial R}{\partial e}$$

Lagrange's planetary equations

3. Orbital perturbations

Variation of parameters

- Perturbative potential must be expressed by orbital elements $\Omega, i, \omega, a, e, M$

$$R = -\frac{\mu R^2}{r^3} J_2 \frac{3\sin^2\psi - 1}{2}$$

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$$R = -\frac{\mu R^2}{r^3} J_2 \frac{3\sin^2\psi - 1}{2}$$

$$\sin\psi = \sin i \sin(\omega + \theta)$$

$$r = \frac{a(1-e^2)}{1+e\cos\theta}$$

2. approximation - ellipsoid

$$R \neq R(\Omega)$$

3. Orbital perturbations

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- average value of R in one period T

$$\overline{R} = \frac{1}{T}\int_0^T R\,dt = \frac{1}{2\pi}\int_0^{2\pi} R\,dM$$

3. Orbital perturbations

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- Perturbative potential must be expressed by orbital elements $\Omega, i, \omega, a, e, M$

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$$\overline{R} = -\frac{1}{4}\frac{\mu R^2}{a^3(1-e^2)^{3/2}} J_2 (3\sin^2 i - 2)$$

3. Orbital perturbations

Variation of parameters

- Perturbative potential can be decomposed into average (secular) and periodic part

$$R = R_s + R_p$$

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average value in one period is zero

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$$R = R_s + R_p$$

$$R_s = \overline{R}$$

average value in one period is zero

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- Perturbative potential can be decomposed into average (secular) and periodic part

$$R = R_s + R_p$$

$$R_s = \overline{R}$$

average value in one period is zero

- Replacing perturbative potential by its secular part

$$R \Rightarrow R_s \Rightarrow \overline{R}$$

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- Perturbative potential can be decomposed into average (secular) and periodic part

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$$R = R(a, e, i)$$

3. Orbital perturbations

Variation of parameters

$$R = R(a, e, i)$$

- Lagrange's planetary equations

$$\frac{de}{dt} = e\left(\frac{\partial R}{\partial \omega}, \frac{\partial R}{\partial M}\right) \qquad \frac{di}{dt} = i\left(\frac{\partial R}{\partial \Omega}, \frac{\partial R}{\partial \omega}\right) \qquad \frac{da}{dt} = a\left(\frac{\partial R}{\partial M}\right)$$

$$\frac{d\,\Omega}{dt} = \Omega\left(\frac{\partial R}{\partial i}\right)$$

$$\frac{dM}{dt} = M\left(\frac{\partial R}{\partial a}, \frac{\partial R}{\partial e}\right)$$

$$\frac{d\,\omega}{dt} = \omega\left(\frac{\partial R}{\partial i}, \frac{\partial R}{\partial e}\right)$$

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e, i, a are constants

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e, i, a are constants

$$\frac{d\,\Omega}{dt} = \Omega\left(\frac{\partial R}{\partial i}\right) \Rightarrow \frac{d\,\Omega}{dt} = -\frac{3}{2(1-e^2)^2} n J_2 \left(\frac{R}{a}\right)^2 \cos i$$

$$\frac{dM}{dt} = M\left(\frac{\partial R}{\partial a}, \frac{\partial R}{\partial e}\right) \Rightarrow \frac{dM}{dt} = n + \frac{3}{4(1-e^2)^{3/2}} n J_2 \left(\frac{R}{a}\right)^2 (3\cos^2 i - 1)$$

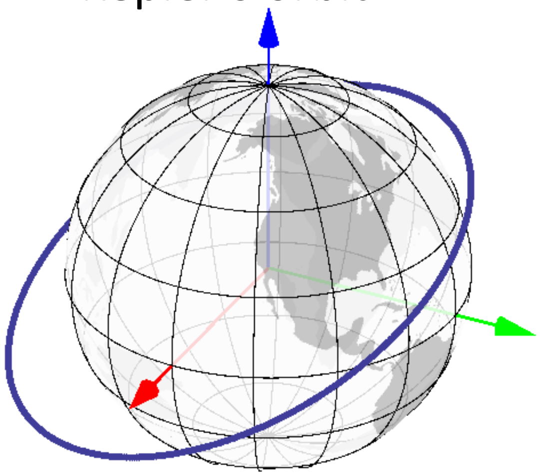
$$\frac{d\,\omega}{dt} = \omega\left(\frac{\partial R}{\partial i}, \frac{\partial R}{\partial e}\right) \Rightarrow \frac{d\,\omega}{dt} = \frac{3}{4(1-e^2)^2} n J_2 \left(\frac{R}{a}\right)^2 (5\cos^2 i - 1)$$

3. Orbital perturbations

Variation of parameters

- Lagrange’s planetary equations

Kepler’s orbit



input parameters:

$$r_p = [2004.75,\ 6174.08,\ 1567.56\ ]\,\text{km}$$

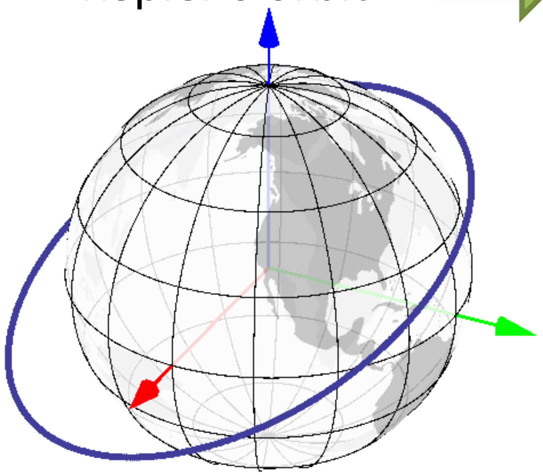
$$v_p = [-7.556,\ 1.581,\ 3.435\ ]\,\text{km/s}$$

3. Orbital perturbations

Variation of parameters

- Lagrange's planetary equations

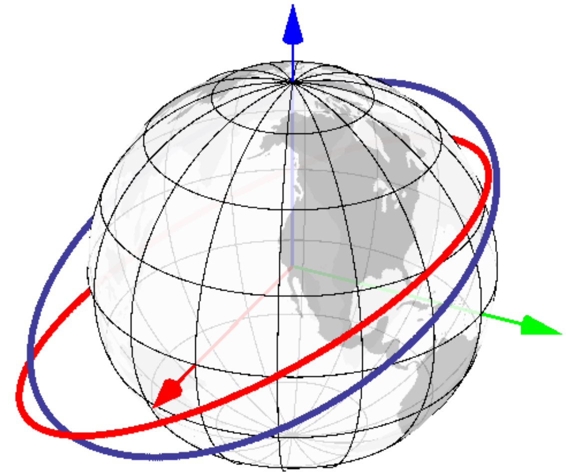
Kepler's orbit

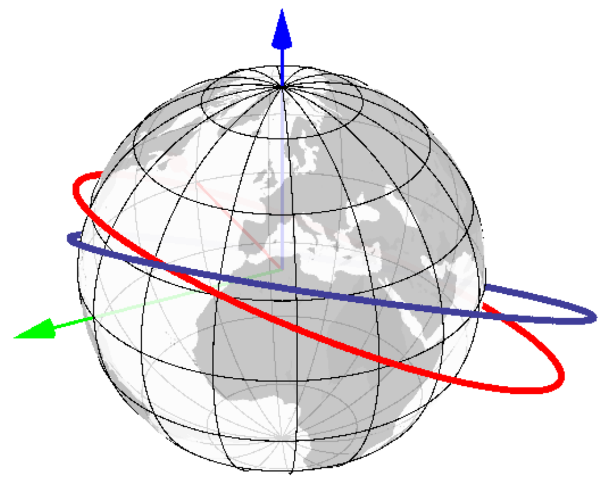


Perturbed orbit
after 100 x T
only J_2 is considered

$$\Delta\Omega = -32.9°$$

$$\Delta\omega = 53.9°$$





input parameters:

$$r_p = \left[2004.75,\ \ 6174.08,\ \ 1567.56\ \right]\mathrm{km}$$

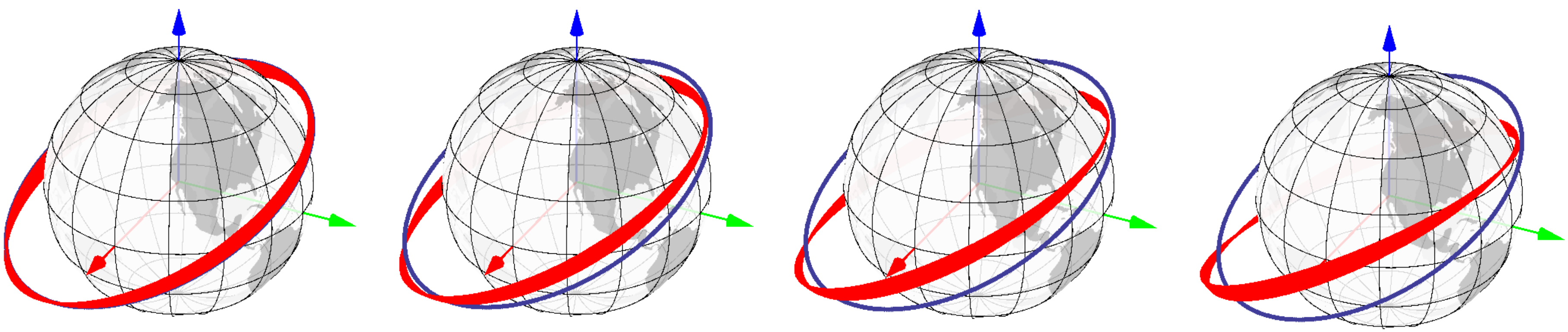
$$v_p = \left[-7.556,\ 1.581,\ 3.435\ \right]\mathrm{km/s}$$

3. Orbital perturbations

Variation of parameters

- Numerical solution of orbital equations

Red color – perturbed orbit in specific time range
Blue color – unperturbed Kepler's orbit



$t = \langle 0, 40T \rangle$ $t = \langle 40T, 80T \rangle$ $t = \langle 80T, 120T \rangle$ $t = \langle 120T, 160T \rangle$

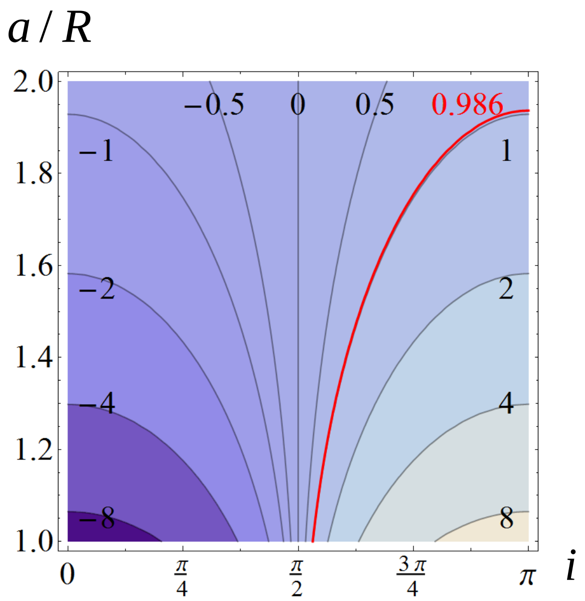
3. Orbital perturbations

Examples of orbits

- Sun-synchronous orbits
  - Earth rotates counterclockwise around the Sun with angular velocity 0.986° per day
  - if satellite orbit rotates clockwise with the same angular velocity, position of orbit relative to the Sun will be still the same

$$\frac{d\Omega}{dt} = -\frac{3}{2\left(1-e^2\right)^2} n J_2 \left(\frac{R}{a}\right)^2 \cos i$$

$$\frac{d\Omega}{dt} = -\frac{3}{2} J_2 \sqrt{\frac{\mu}{R^3}} \left(\frac{R}{a}\right)^{7/2} \cos i$$



3. Orbital perturbations

Examples of orbits

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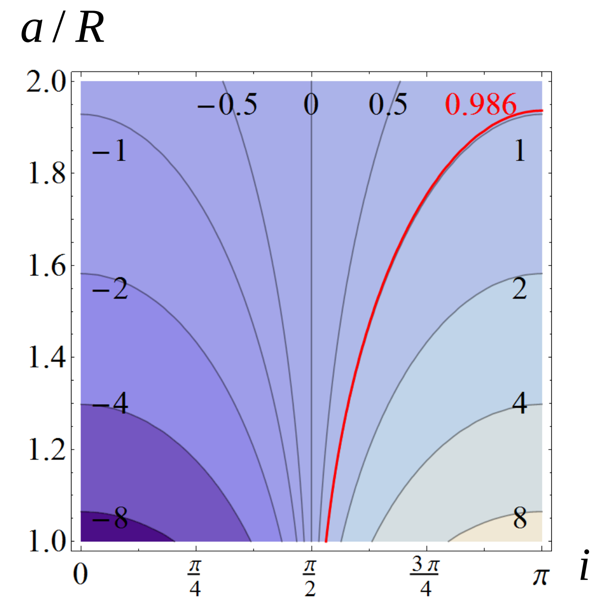
  - Earth rotates counterclockwise around the Sun with angular velocity 0.986° per day
  - if satellite orbit rotates clockwise with the same angular velocity, position of orbit relative to the Sun will be still the same

$$i_{\min} = 95.6° \text{ for } a = R$$

$$a_{\max} = 12331 \text{ km for } i = 180°$$



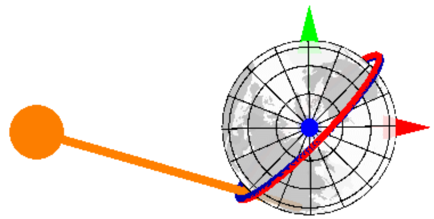
Operating S-s satellites: $h = 700 - 900$ km
orbit: circular or near circular

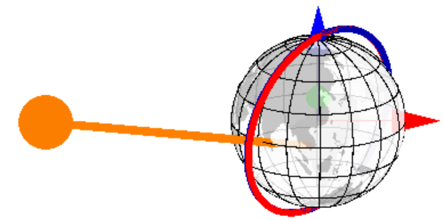
3. Orbital perturbations

Examples of orbits

- Sun-synchronous orbits

$$\text{Landsat}-4:\ i = 99.07^\circ$$
$$a = 7285.799\ \text{km}$$





view from Sun

Blue: Kepler's orbit
Red: sun-synchronous orbit
Orange: Sun and sun beam

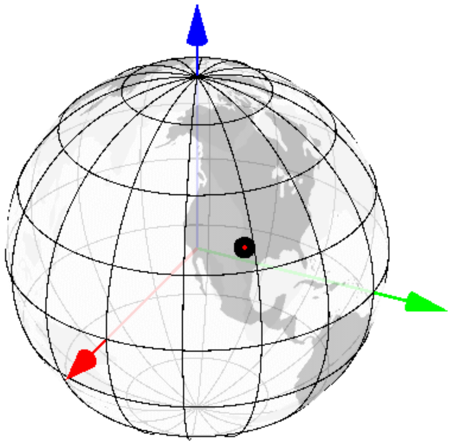
3. Orbital perturbations

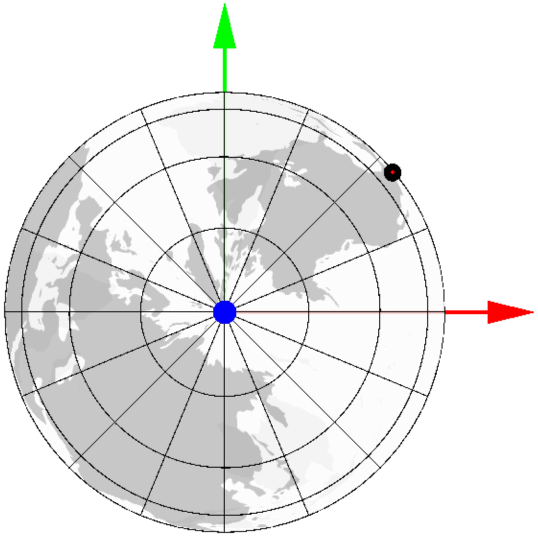
Examples of orbits

- Sun-synchronous orbits

Landsat – 4: $i = 99.07°$

$a = 7285.799\,\text{km}$





view from Earth

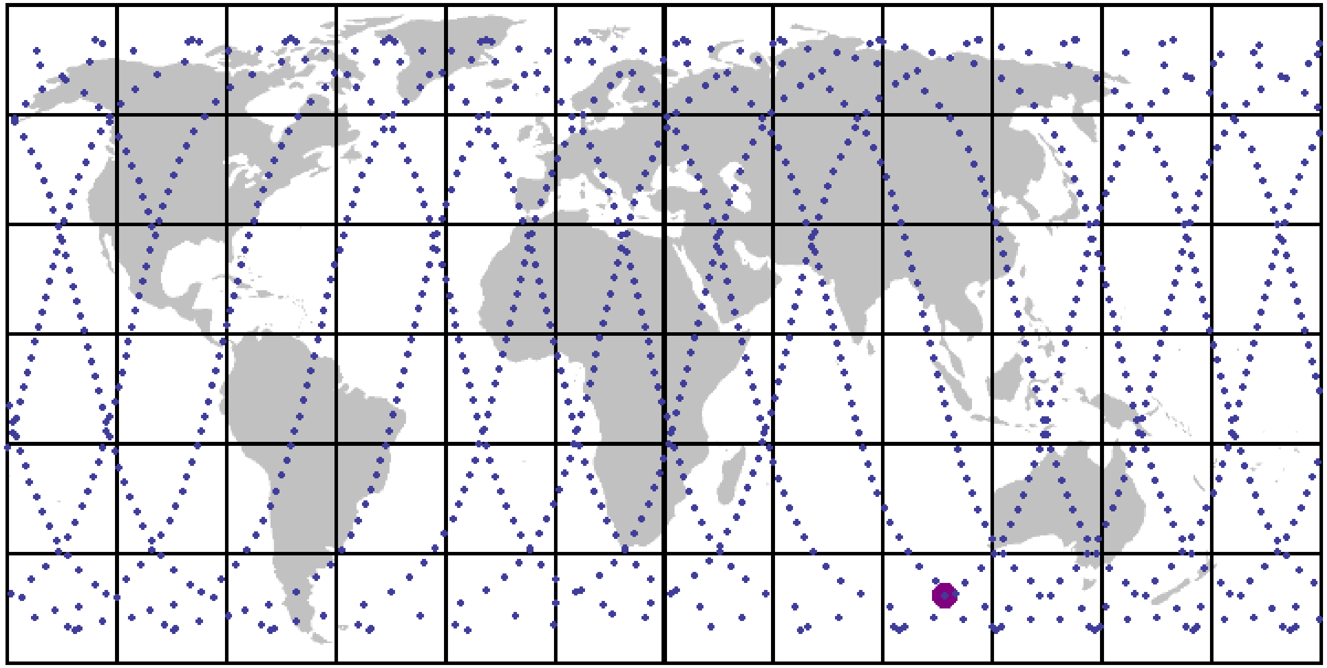
3. Orbital perturbations

Examples of orbits

- Sun-synchronous orbits

Landsat – 4: $i = 99.07°$

$a = 7285.799 \text{ km}$



4. Orbital maneuvers

- Impulsive maneuvers
- Hohmann transfer
- Non-Hohmann transfer
- Plane change maneuvers

4. Orbital maneuvers

Impulsive maneuvers

- brief firings of rocket motors change the magnitude and direction of the velocity vector instantaneously
- during an impulsive maneuver, the position of the spacecraft is considered to be fixed, only the velocity changes → impulsive maneuver is an idealization
- velocity increment is related to consumed propellant

$$\Delta v_S = -u_e \mathrm{Ln}\left(\frac{m_{S0} - \Delta m_S}{m_{S0}}\right)$$

4. Orbital maneuvers

Impulsive maneuvers

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$$u_e = I_s g$$

$$\Delta v_S = -u_e \mathrm{Ln}\left(\frac{m_{S0} - \Delta m_S}{m_{S0}}\right)$$

4. Orbital maneuvers

Impulsive maneuvers

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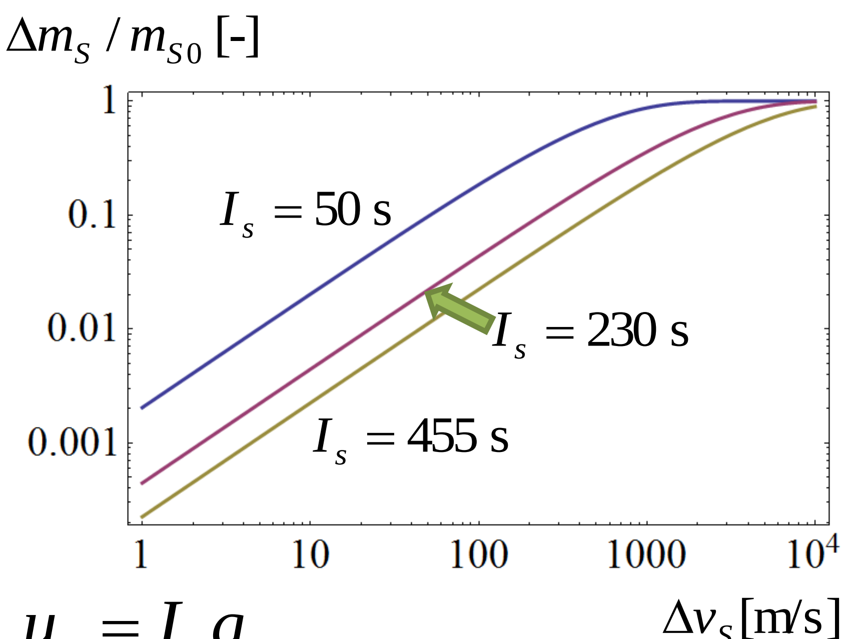
→

$$\frac{\Delta m_S}{m_{S0}} = 1 - e^{-\frac{\Delta v_S}{I_S g}}$$

4. Orbital maneuvers

Impulsive maneuvers

- specific impulse characteristics



Propellant	Specific impulse I_s [s]
cold gas	50
Monopropellant hydrazine	230
LOX/LH2	455
Ion propulsion	>3000

$$u_e = I_s g$$

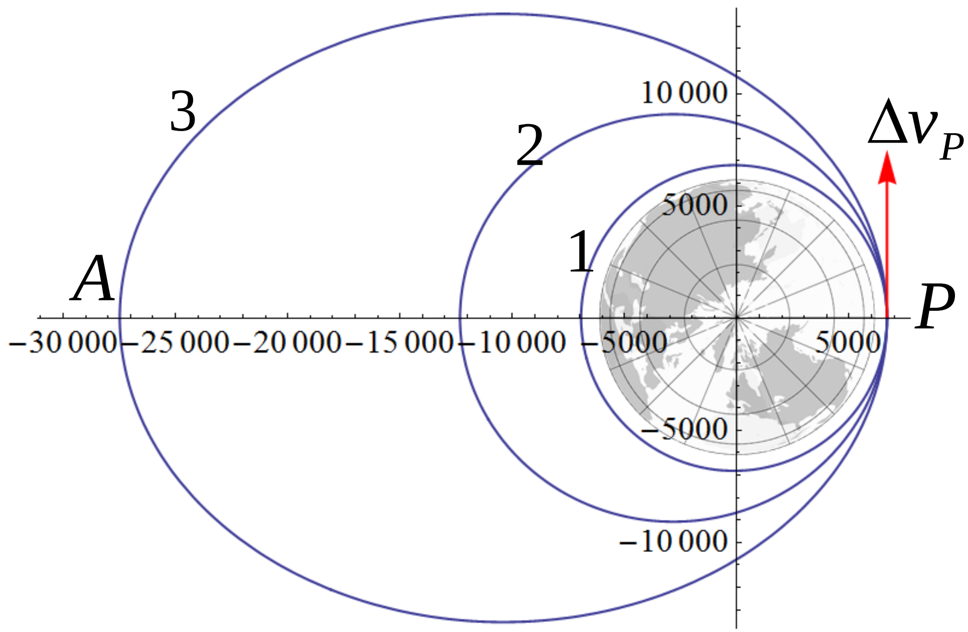
$$\Delta v_S = -u_e \mathrm{Ln}\left(\frac{m_{S0} - \Delta m_S}{m_{S0}}\right)$$

$$\frac{\Delta m_S}{m_{S0}} = 1 - e^{-\frac{\Delta v_S}{I_S g}}$$

4. Orbital maneuvers

Impulsive maneuvers

- impulse at periapsis



$$r_P = r_{\text{Earth}} + 300\,\text{km}$$

$$v_{P1} = 7.8\,\text{km/s}$$

$$e_1 = 0.019$$

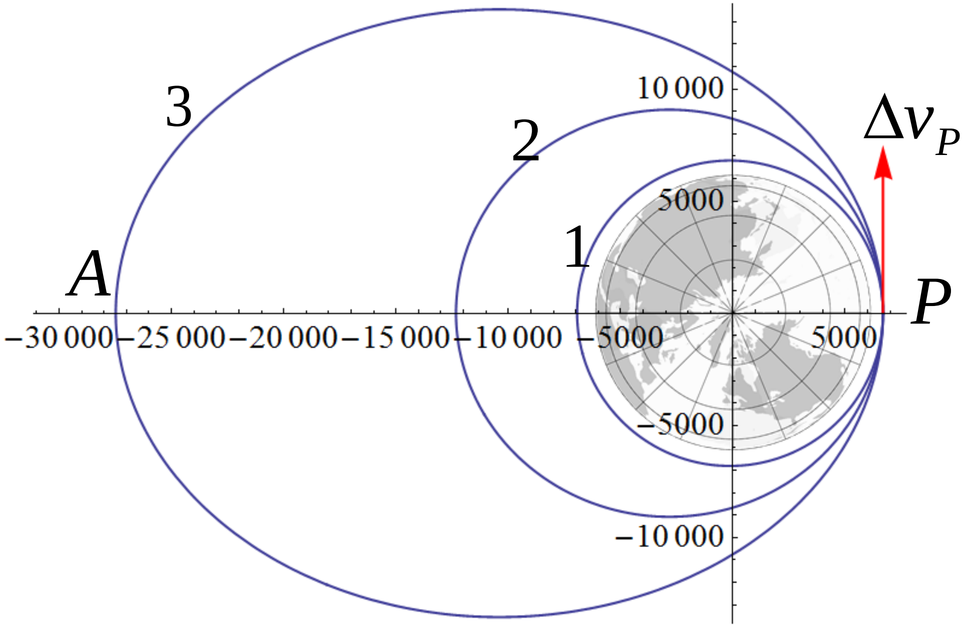
r_P, v_{P1}

Δv_P

4. Orbital maneuvers

Impulsive maneuvers

- impulse at periapsis



$$r_P = r_{\text{Earth}} + 300\,\text{km}$$

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$$e_1 = 0.019$$

r_P, v_{P1}

Δv_P

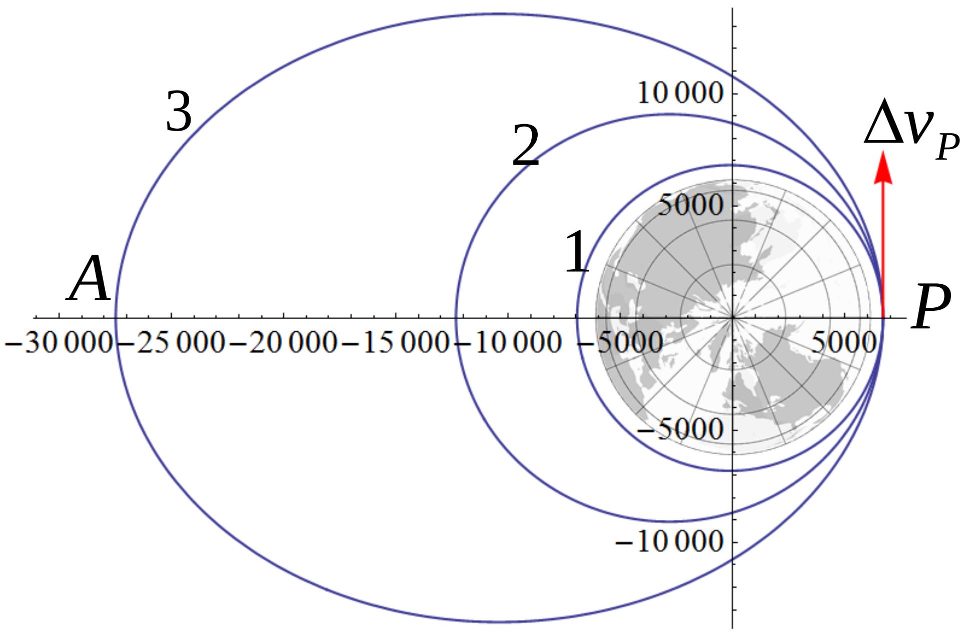
$$v_{P2} = v_{P1} + \Delta v_P$$

4. Orbital maneuvers

Impulsive maneuvers

- impulse at periapsis

$$r_P = r_{\text{Earth}} + 300\,\text{km}$$
$$v_{P1} = 7.8\,\text{km/s}$$
$$e_1 = 0.019$$



r_P, v_{P1}

Δv_P

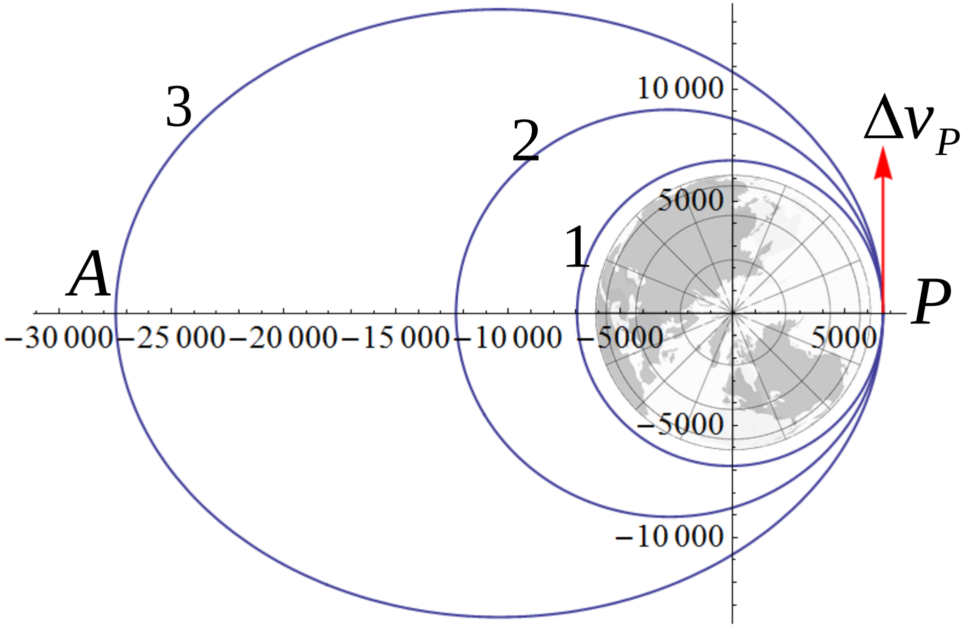
$$v_{P2} = v_{P1} + \Delta v_P$$

$$h_2 = r_P v_{P2}$$

4. Orbital maneuvers

Impulsive maneuvers

- impulse at periapsis



$$r_P = r_{\text{Earth}} + 300\,\text{km}$$

$$v_{P1} = 7.8\,\text{km/s}$$

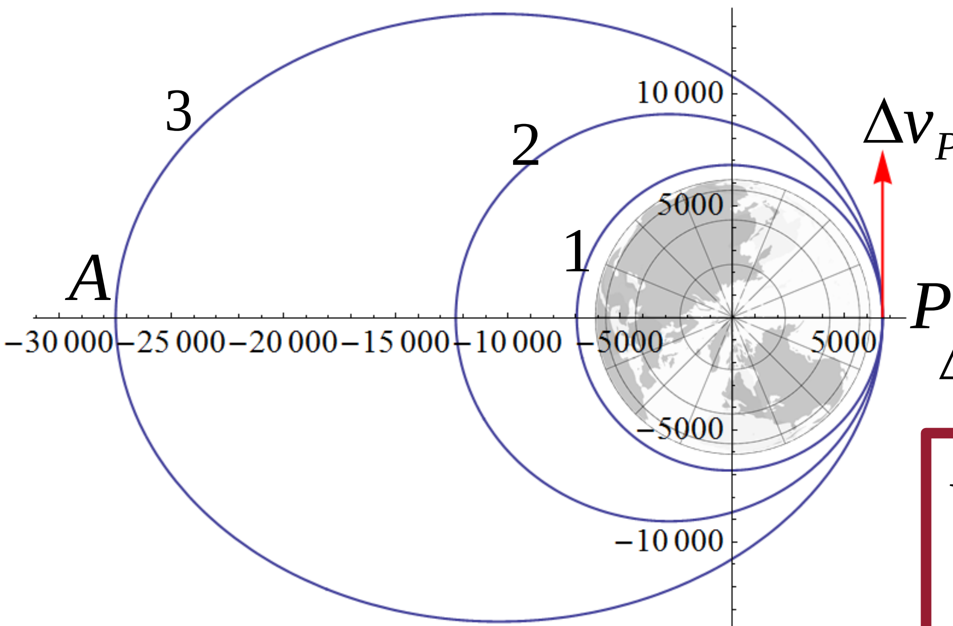
$$e_1 = 0.019$$

r_P, v_{P1} and Δv_P → $v_{P2} = v_{P1} + \Delta v_P$ → $h_2 = r_P v_{P2}$ → e_2 → **new orbit**

4. Orbital maneuvers

Impulsive maneuvers

- impulse at periapsis



$$r_P = r_{\text{Earth}} + 300\,\text{km}$$

$$v_{P1} = 7.8\text{km/s}$$

$$e_1 = 0.019$$

$$\Delta v_{P2} = 1\text{km/s}$$

$$\Delta v_{P3} = 2\text{km/s}$$

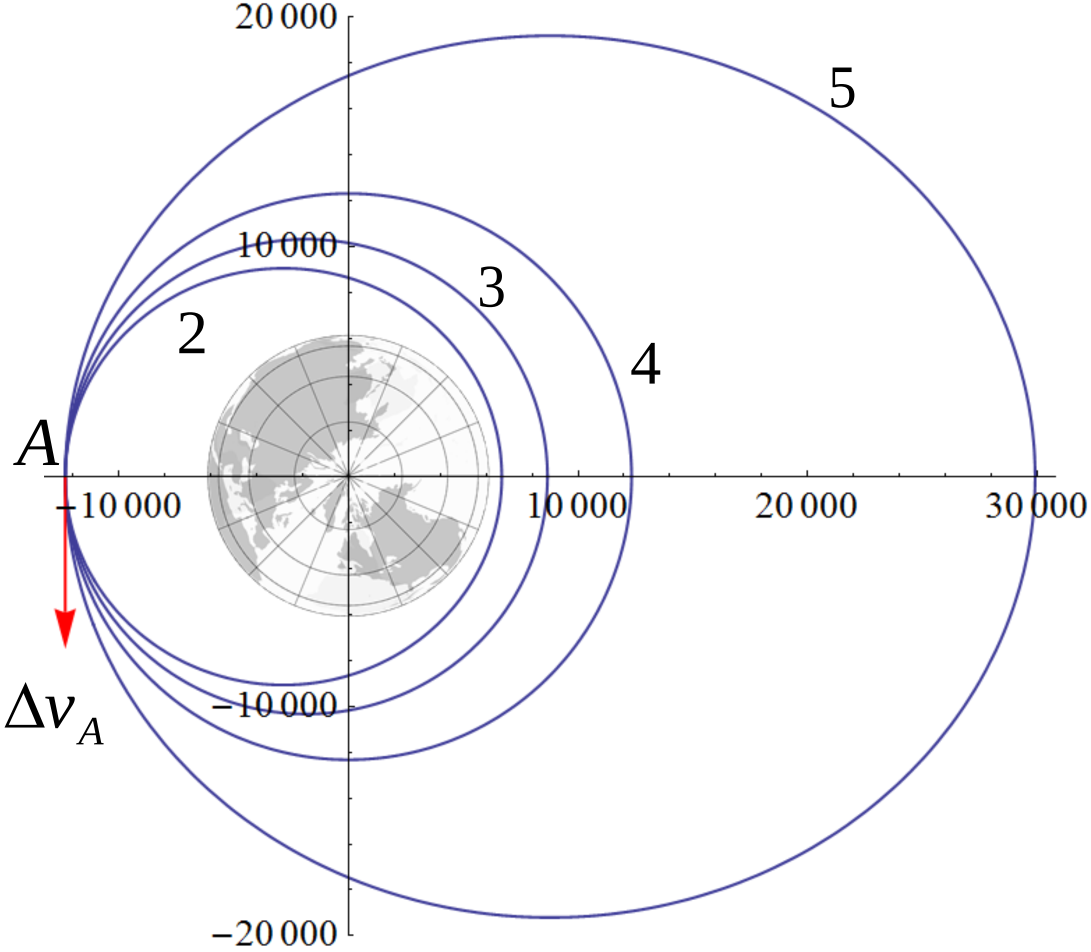
$v_{P2} = 8.8\text{km/s}$	$v_{P3} = 9.8\text{km/s}$
$e_2 = 0.297$	$e_3 = 0.609$
$r_{A3} = 12331.4\text{km}$	$r_{A3} = 27482.1\text{km}$

r_P, v_{P1} and Δv_P → $v_{P2} = v_{P1} + \Delta v_P$ → $h_2 = r_P v_{P2}$ → e_2 → **new orbit**

4. Orbital maneuvers

Impulsive maneuvers

- impulse at apoapsis



$$r_P = r_{\text{Earth}} + 300\,\text{km}$$

$$v_{P2} = 8.8\text{km/s}$$

$$e_2 = 0.297$$

$$\Delta v_{A3} = 0.4\text{km/s}$$

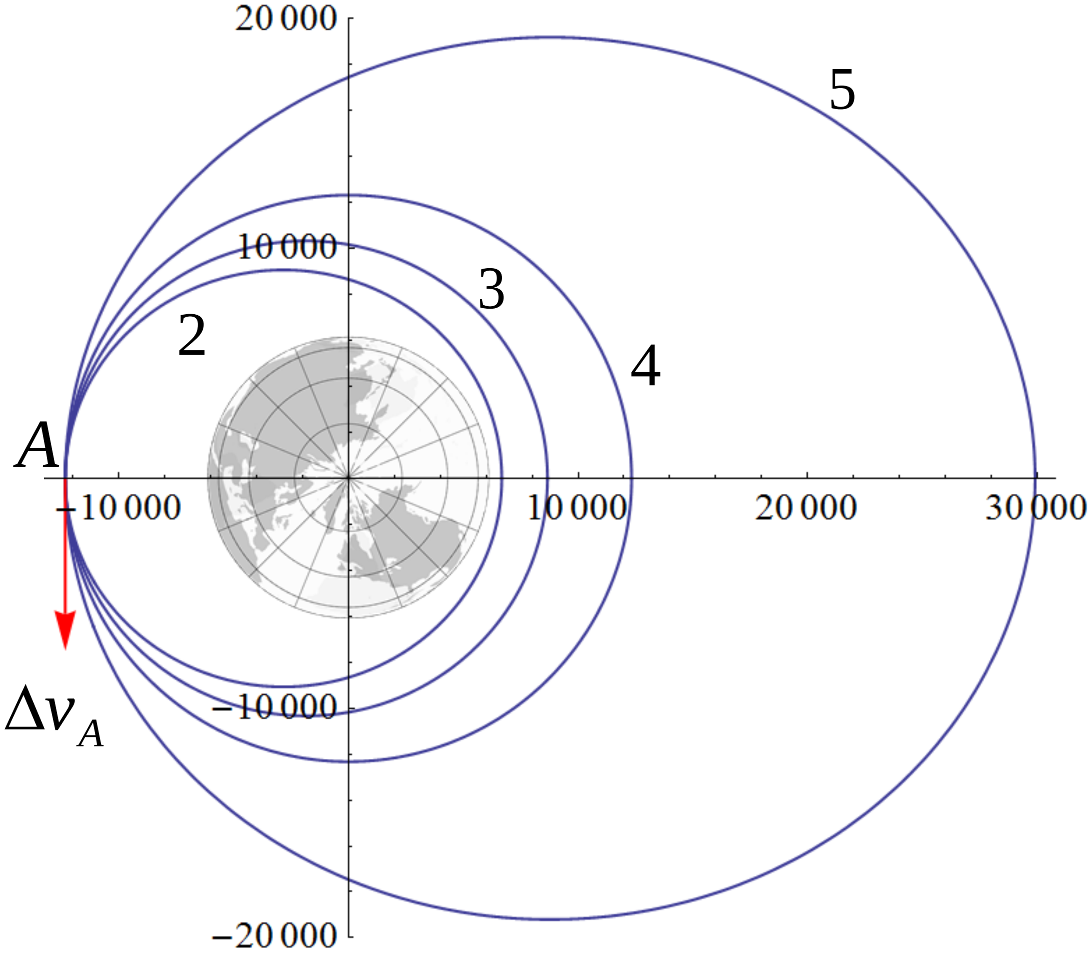
$$\Delta v_{A4} = 0.919\,\text{km/s}$$

$$\Delta v_{A5} = 2.\text{km/s}$$

4. Orbital maneuvers

Impulsive maneuvers

- impulse at apoapsis



$$r_P = r_{\text{Earth}} + 300\,\text{km}$$

$$v_{P2} = 8.8\,\text{km/s}$$

$$e_2 = 0.297$$

$$\Delta v_{A3} = 0.4\,\text{km/s}$$

$$\Delta v_{A4} = 0.919\,\text{km/s}$$

$$\Delta v_{A5} = 2.\,\text{km/s}$$

$$e_3 = 0.174$$

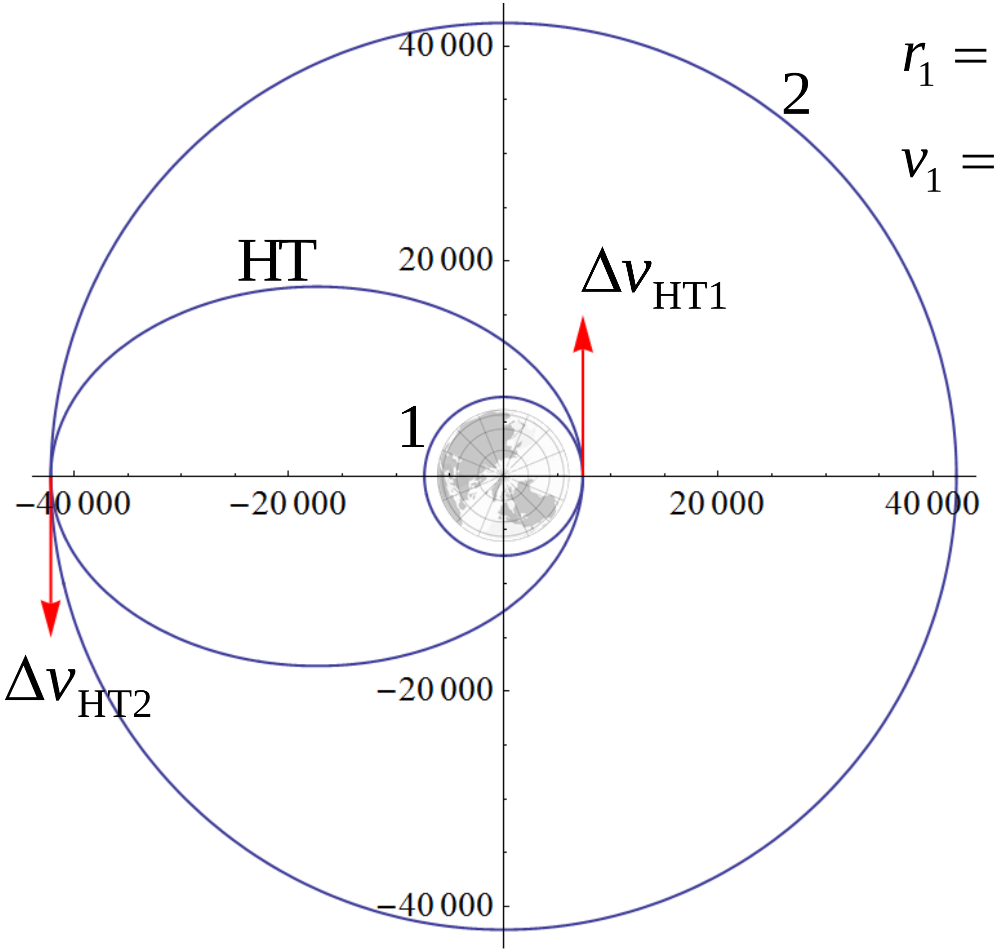
$$e_4 = 0 \Rightarrow \text{circle}$$

$$e_5 = 0.416 \Rightarrow A \Leftrightarrow P$$

4. Orbital maneuvers

Hohmann transfer

- 2 impulse maneuvers



circle orbit 1:

$$r_1 = r_{\text{Earth}} + 1000\text{ km}$$

$$v_1 = \sqrt{\mu / r_1}$$

circle orbit 2 - GEO:

$$r_2 = r_{\text{GEO}} = 42164\text{ km}$$

$$v_2 = \sqrt{\mu / r_1}$$

Hohmann transfer:

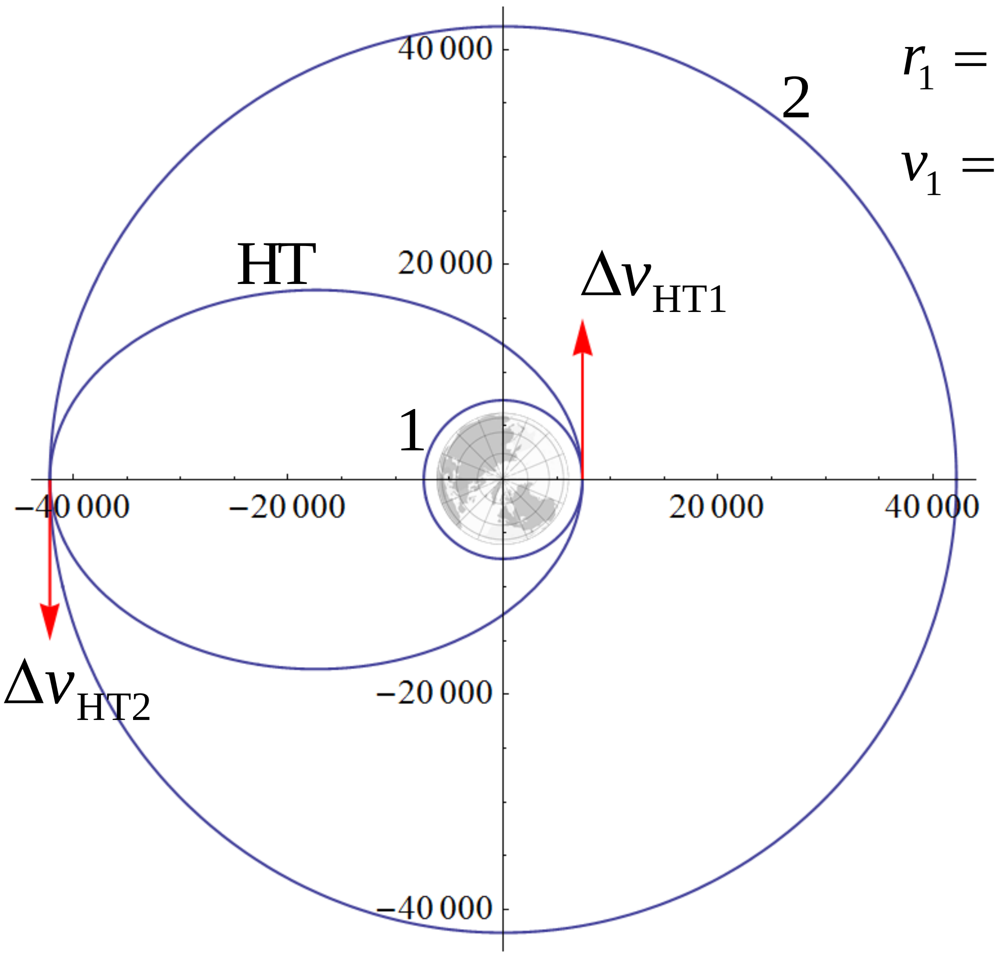
r_1

r_2

4. Orbital maneuvers

Hohmann transfer

- 2 impulse maneuvers



circle orbit 1:

$$r_1 = r_{\mathrm{Earth}} + 1000\ \mathrm{km}$$

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circle orbit 2 - GEO:

$$r_2 = r_{\mathrm{GEO}} = 42164\ \mathrm{km}$$

$$v_2 = \sqrt{\mu / r_1}$$

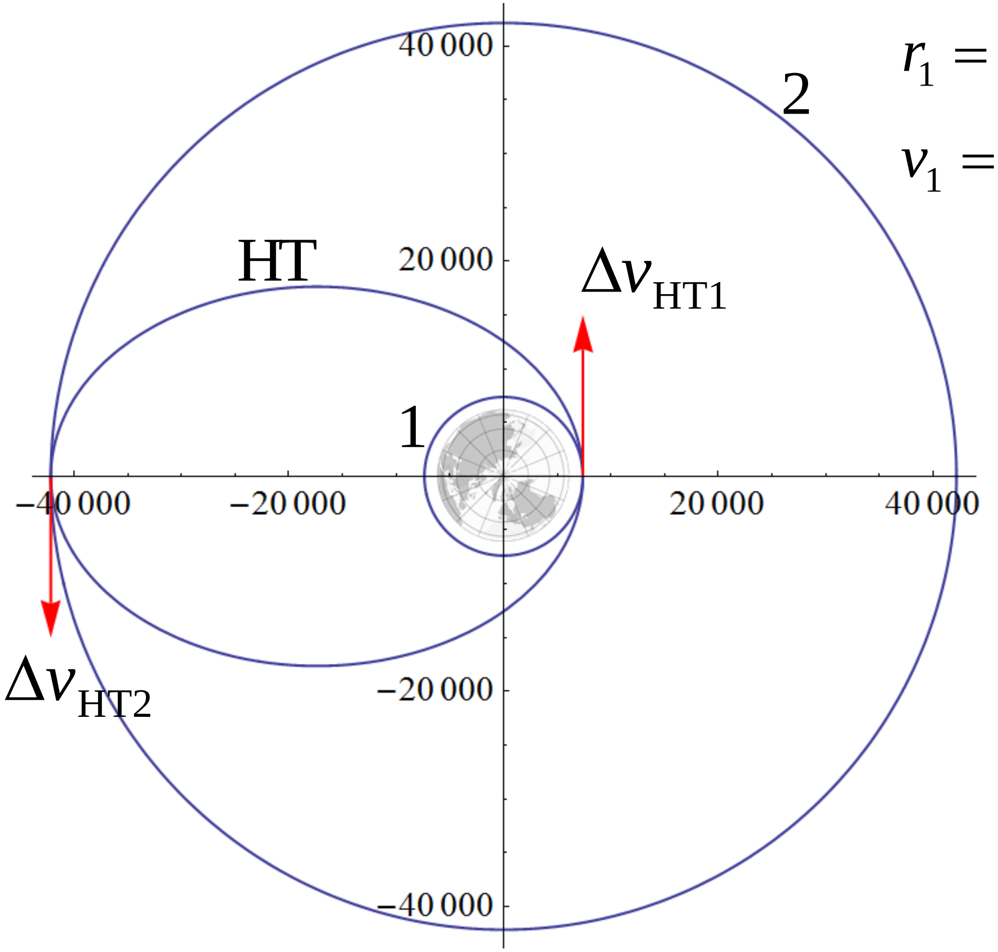
Hohmann transfer:

r_1, r_2 → a_{HT}, e_{HT} → h_{HT}

4. Orbital maneuvers

Hohmann transfer

- 2 impulse maneuvers



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circle orbit 2 - GEO:

$$r_2 = r_{\text{GEO}} = 42164\,\text{km}$$

$$v_2 = \sqrt{\mu / r_1}$$

Hohmann transfer:

r_1, r_2 → a_{HT}, e_{HT} → h_{HT}

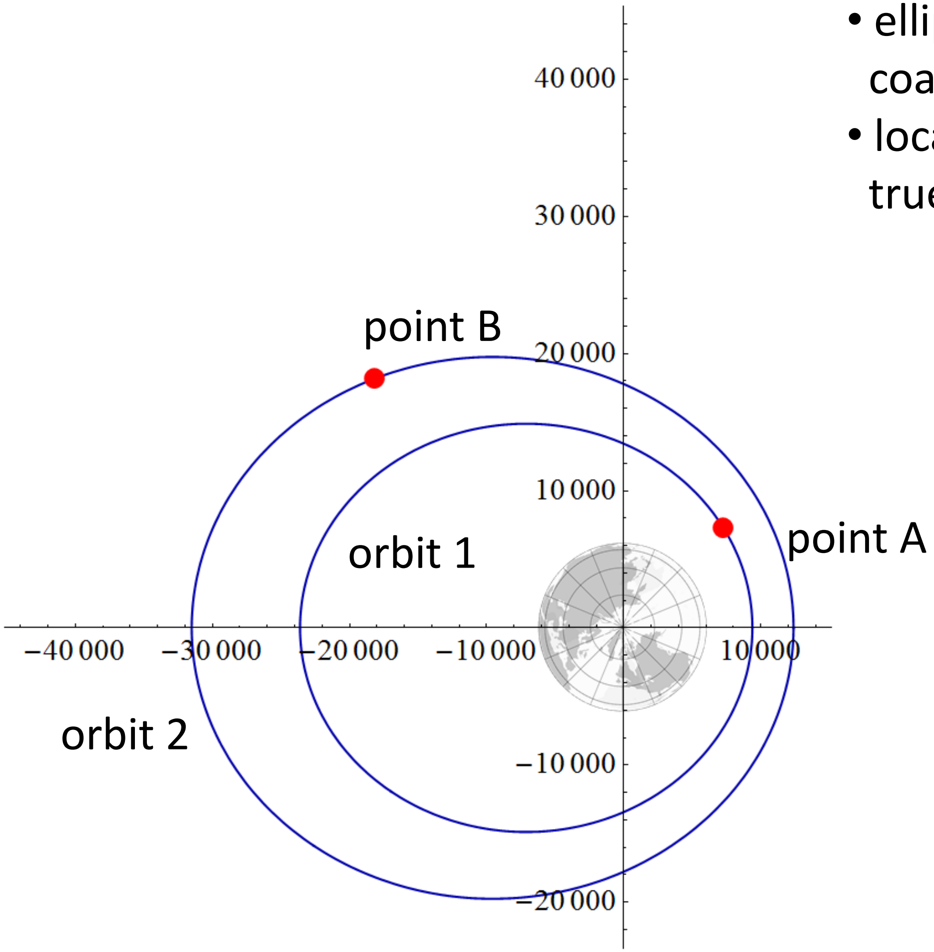
$$\Delta v_{\text{HT1}} = 2.24\,\text{km/s}$$

$$\Delta v_{\text{HT2}} = 1.39\,\text{km/s}$$

4. Orbital maneuvers

Non-Hohmann transfer

- 2 impulse maneuvers

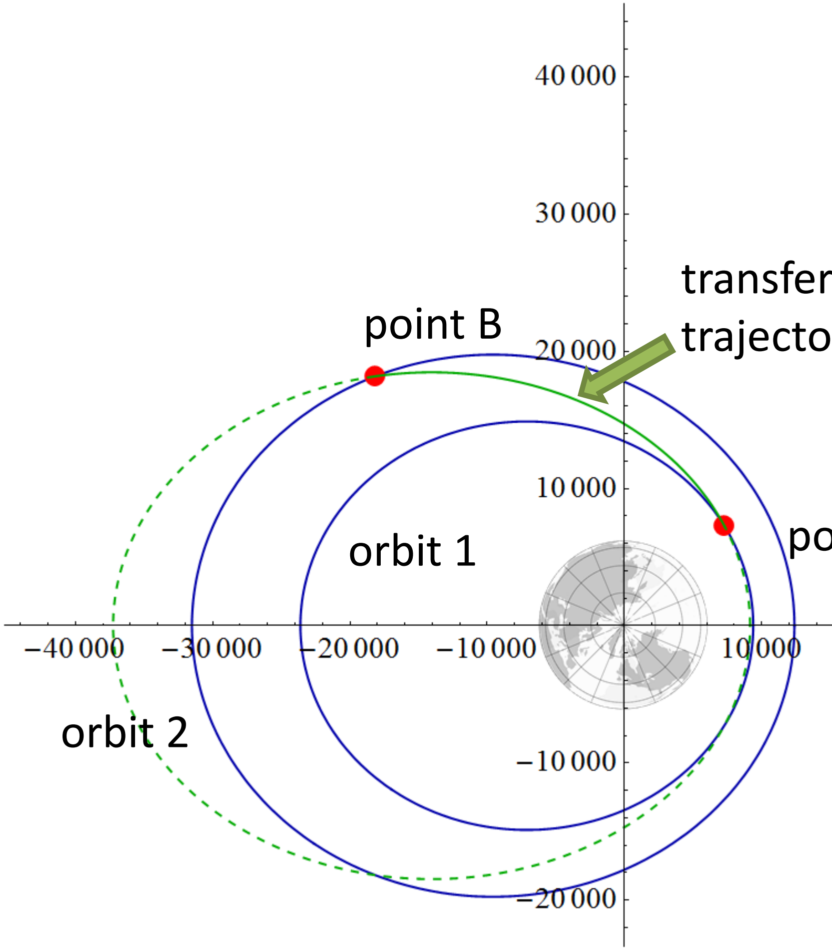


- transfer between 2 elliptical orbits
- elliptical orbits are in the same plane and are coaxial
- locations of points A and B are defined by true anomaly θ_A a θ_B

4. Orbital maneuvers

Non-Hohmann transfer

- 2 impulse maneuvers



- transfer between 2 elliptical orbits
- elliptical orbits are in the same plane and are coaxial
- locations of points A and B are defined by true anomaly θ_A a θ_B

transfer trajectory

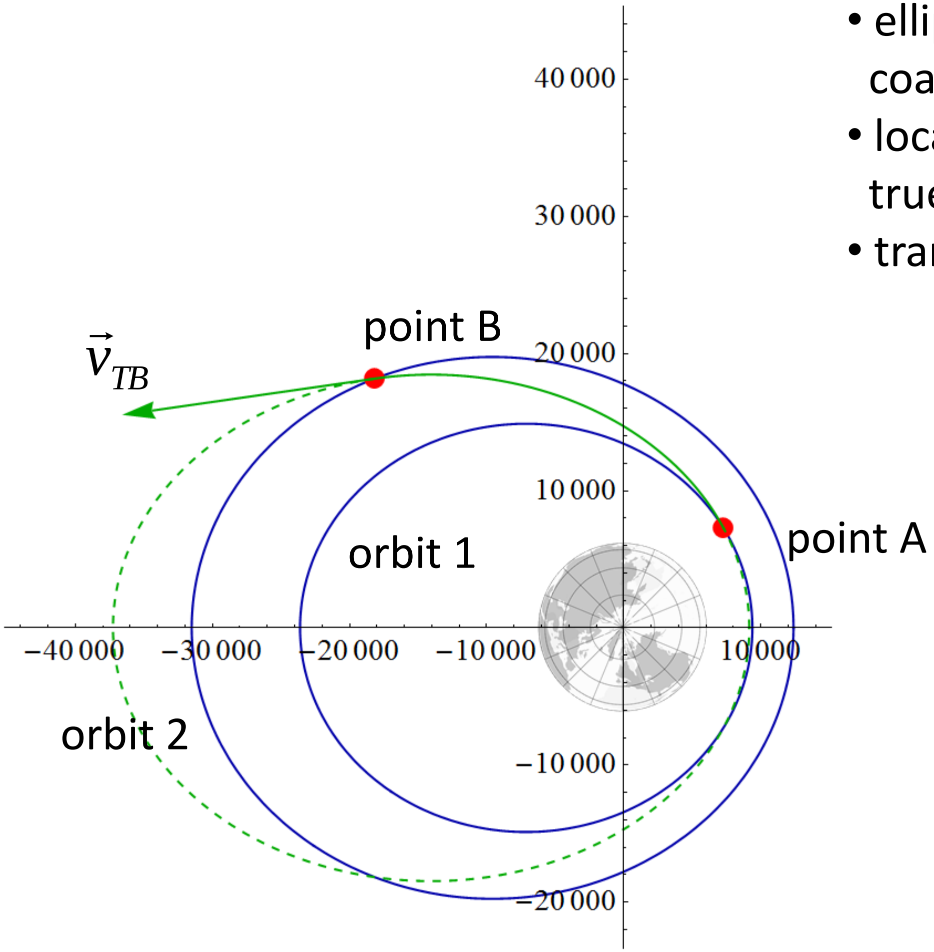
$$r_A = \frac{h^2}{\mu}\frac{1}{(1+e\cos\theta_A)} \qquad r_B = \frac{h^2}{\mu}\frac{1}{(1+e\cos\theta_B)}$$

h, e

4. Orbital maneuvers

Non-Hohmann transfer

- 2 impulse maneuvers



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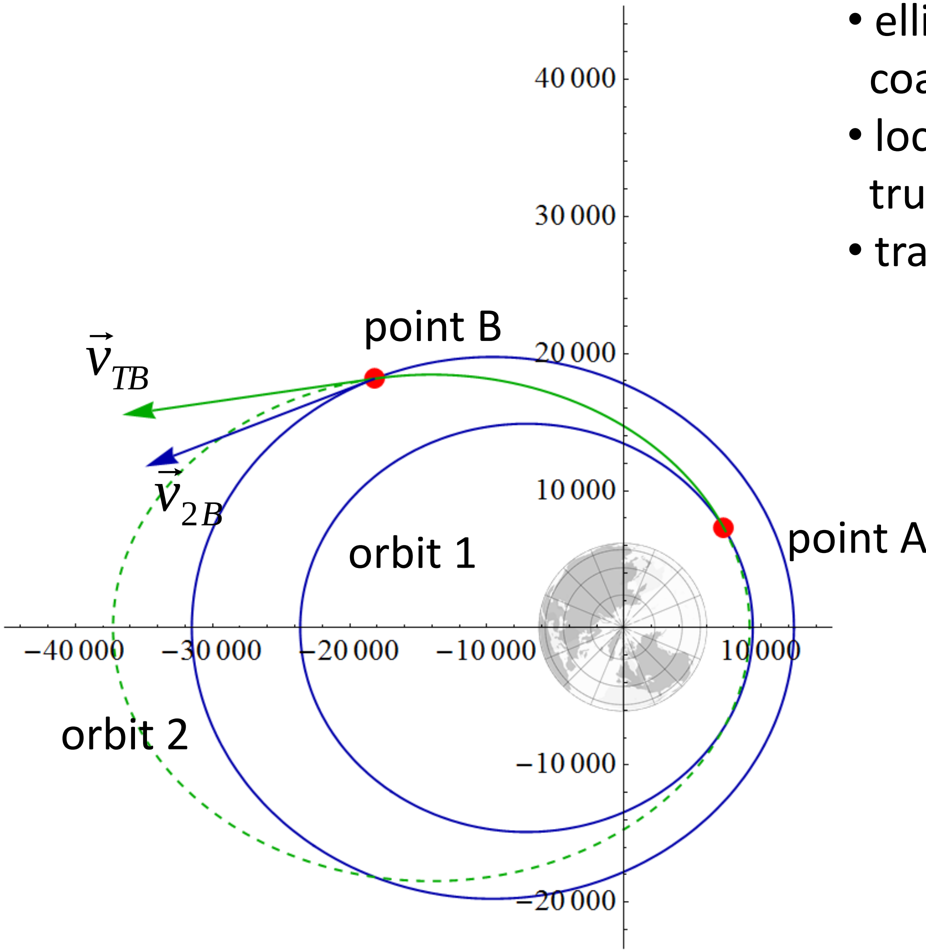
- transfer orbit h, e

$\vec{v}_{TB}$

4. Orbital maneuvers

Non-Hohmann transfer

- 2 impulse maneuvers



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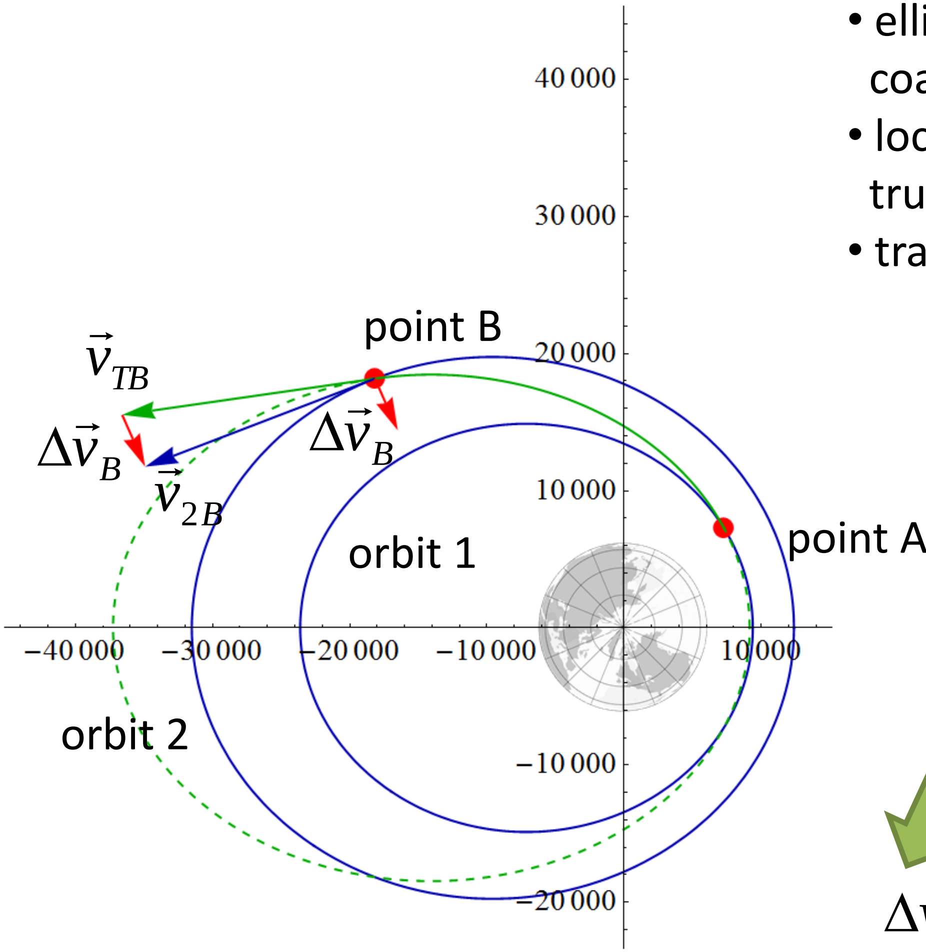
- transfer orbit h, e

$\vec{v}_{TB}$ $\vec{v}_{2B}$

4. Orbital maneuvers

Non-Hohmann transfer

- 2 impulse maneuvers



- transfer between 2 elliptical orbits
- elliptical orbits are in the same plane and are coaxial
- locations of points A and B are defined by true anomaly θ_A a θ_B
- transfer orbit h, e

$\vec{v}_{TB}$ $\vec{v}_{2B}$

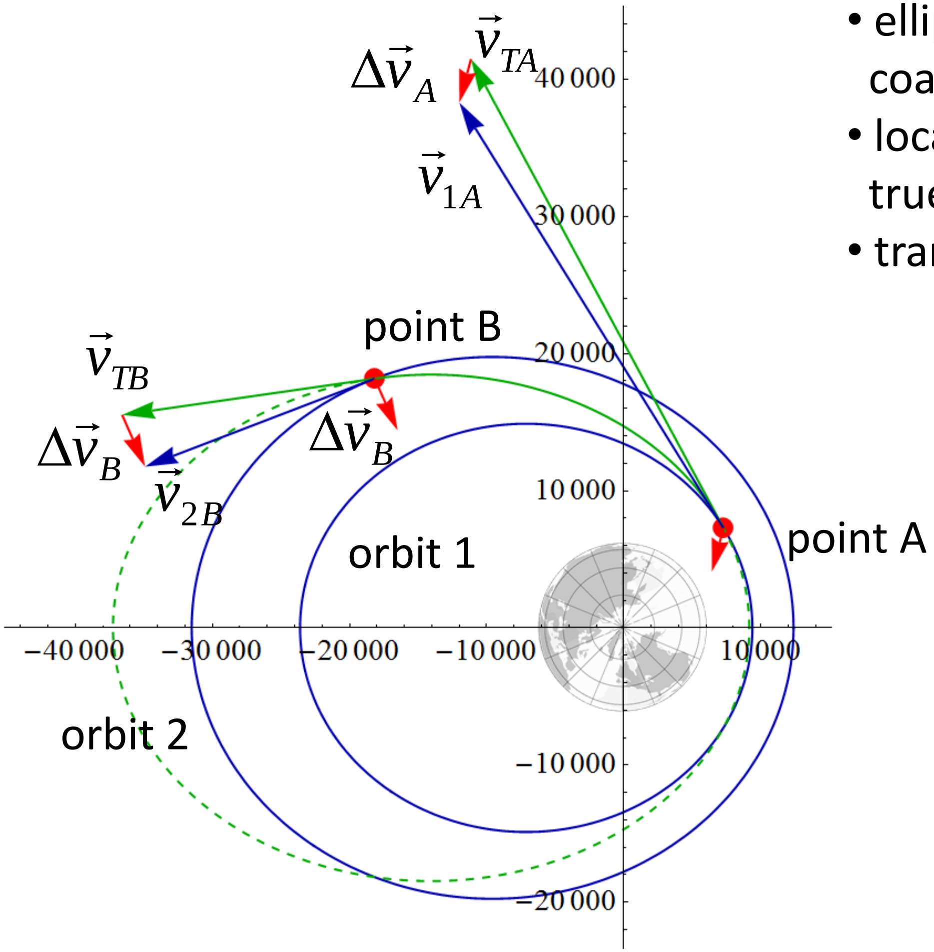
$$\Delta v_B = \sqrt{(\vec{v}_{2B} - \vec{v}_{TB}).(\vec{v}_{2B} - \vec{v}_{TB})}$$

$$\Delta v_B = \sqrt{\vec{v}_{2B}\vec{v}_{2B} + \vec{v}_{TB}\vec{v}_{TB} - 2\vec{v}_{2B}\vec{v}_{TB}}$$

4. Orbital maneuvers

Non-Hohmann transfer

- 2 impulse maneuvers



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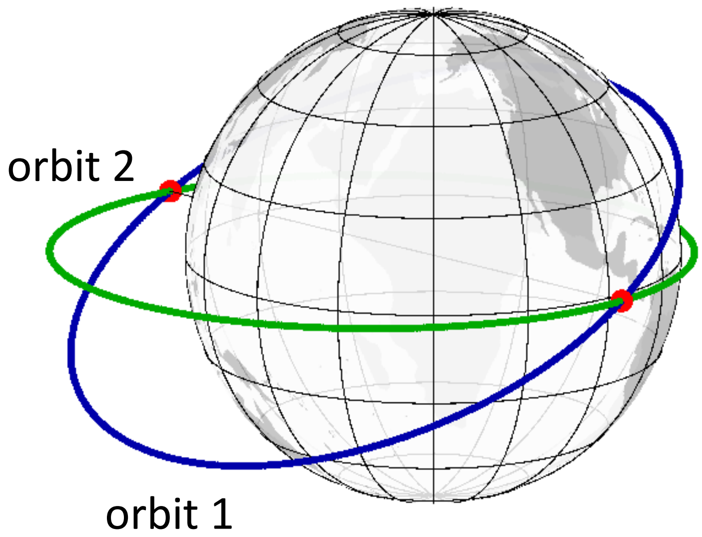
- elliptical orbits are in the same plane and are coaxial
- locations of points A and B are defined by true anomaly θ_A a θ_B
- transfer orbit h, e

4. Orbital maneuvers

Plane change maneuvers

- single impulse maneuver

- To change the orientation of a satellite's orbital plane, typically the inclination, the direction of the velocity vector has to be changed.



4. Orbital maneuvers

Plane change maneuvers

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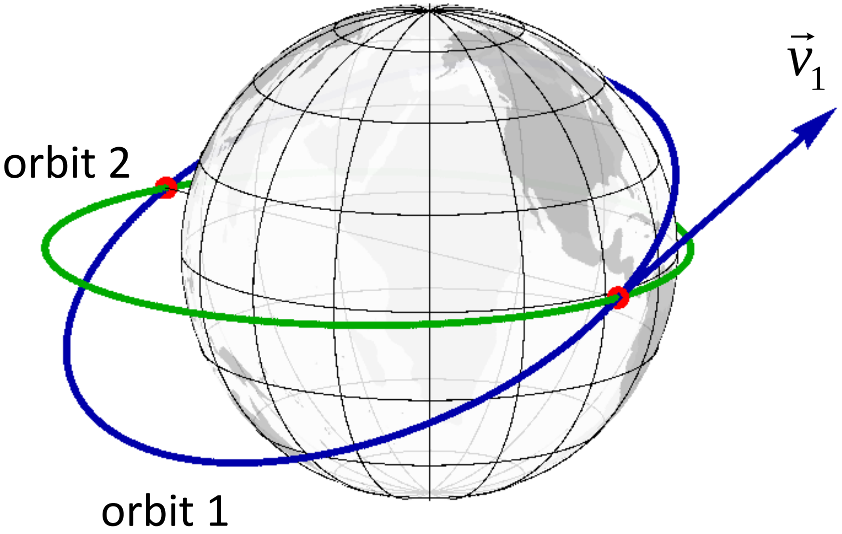

4. Orbital maneuvers

Plane change maneuvers

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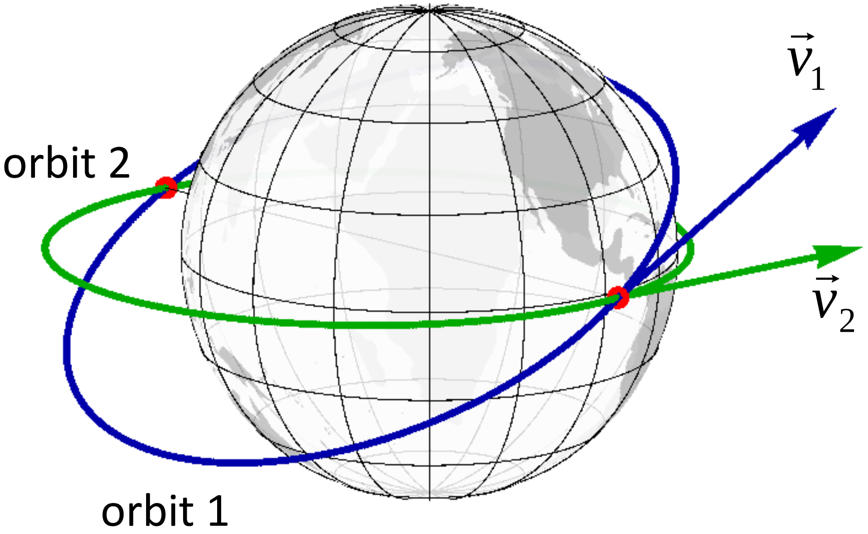

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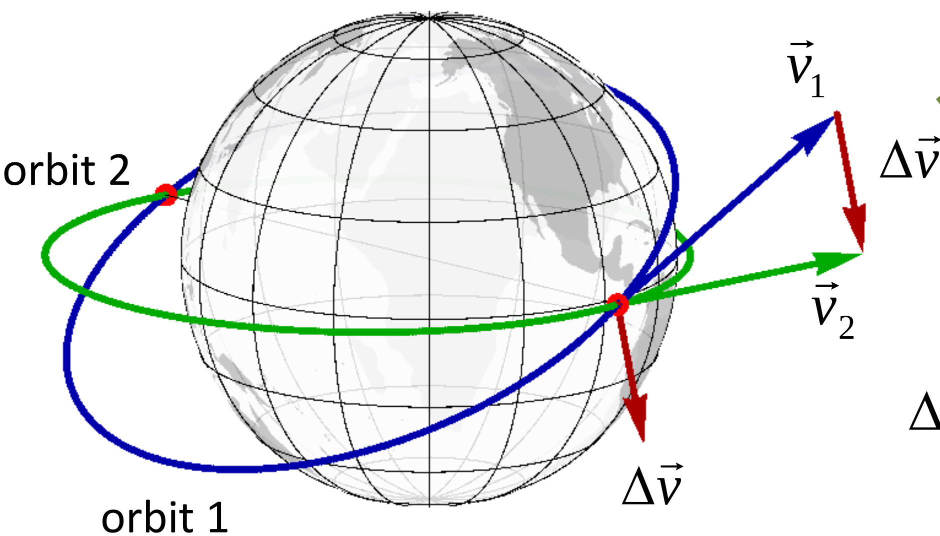

$$\Delta\vec{v} = \vec{v}_2 - \vec{v}_1$$

pure rotation

$$\Delta v = 2\,v\,\sin\left(\frac{\delta}{2}\right)$$

δ is angle between the planes